HOMEWORK 2, MATH 113

1. Let X be an n-point set of d-length sequences $\{x_k\}$ of 0s and 1s, and equip this set with the l_1 size: that is, the distance between two sequences $\{x_k\}$ and $\{y_k\}$ is

$$\sum_{k=1}^{d} |x_k - y_k|.$$

Let's name this metric space l_1^d. Find an isometric embedding of l_1^d into the sequence space $l_\infty^{d'}$, where $d' = 2^d$ and the distance between $\{w_j\}$ and $\{z_j\}$ is the $\sup|w_j - z_j|$.

Hint: Define, for every choice of $\sigma \in \{+1, -1\}^d$, a mapping f into the space of sequences indexed by σ.

2. Review of sequences: p. 51, #11, 17.

3. Let $p > 1$. If X is the space of all sequences $x = \{x_k\}$ such that

$$\|x_k\|_p := (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} < \infty,$$

then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

There were two hints in the lecture: one involved the splitting

$$|x_k + y_k|^p \leq |x_k|(|x_k + y_k|)^{p-1} + |y_k|(|x_k + y_k|)^{p-1},$$

and the other involved using the inequality

$$ab \leq a^p/p + b^q/q,$$

for $q = p/(p-1)$ and $a, b > 0$. The remaining trick involves finding the right normalization...

4.In a metric space (X, d), define the ball centered at x of radius r by $B(x, r) := \{y : d(x, y) < r\}$. A subset A of a metric space is said to be open if for all $x \in A$, there exists an r such that $B(x, r) \subset A$.
(i) Show that these balls are in fact open sets.
(ii) Show that a countable union of open sets is open.
(iii) Show that a finite intersection of open sets is open, but a countable intersection may fail to be open.

5. A sequence in a metric space is called Cauchy if for all n, m, the distance $d(x_n, x_m) \to 0$ as $n, m \to \infty$. A metric space is called complete if every Cauchy sequence converges. By Theorem 2.19, $\mathbb{R}$ is complete.
(i) Show that the sequence space l_1^d of problem (1) is complete.
(ii) Show that the space of all bounded functions $f : X \to \mathbb{R}$, where X is some set, with the metric $d(f, g) := \sup\{|f(x) - g(x)| : x \in X\}$ is complete.