Solution HW(3)

6. Let $\{U_n\}$ countable base of X.

$C = \{x \in X : \forall \text{ open } U \subset X, x \in U : U \cap E \text{ uncountable}\}$

will show $C^c = \{x \in X : \exists \text{ open } U \subset X, x \in U : U \cap E \text{ countable}\}$ open

Suppose $x \in C^c$, then $\exists$ open $U \subset X$: $x \in U$, $U \cap E$ countable

Since U open, $U = \bigcup_{i \in A_x} U_i$,

~~Since~~ And $U \cap E = \bigcup_{i \in A_x} (U_i \cap E)$ countable, $\Rightarrow$ $U_i \cap E$ countable for all $i \in A_x$

Let $A = \{n : U_n \cap E \text{ countable}\}$.

we must have $x \in U \subseteq \bigcup_{n \in A} (U_n \cap E) \subseteq \bigcup_{n \in A} U_n \triangleq G$

$\therefore C^c \subseteq G$

Conversely, for $x \in G = \bigcup_{n \in A} U_n$, $x \in$ some U_n, with property $U_n \cap E$ countable

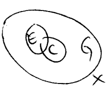
$\therefore x \notin C \Rightarrow x \in C^c$

$\Rightarrow G \subseteq C^c$

$\therefore C^c = G = \underbrace{\bigcup_{n \in A} U_n}$ $\longrightarrow$ open $\therefore C$ is closed

~~Since C is closed, $\bar{E} = C$, $\therefore$ every point of C is a limit point of C.~~

And $E \setminus C = E \cap G = \bigcup_{n \in A} (E \cap U_n)$ each $E \cap U_n$ countable



$\therefore E \setminus C$ is also countable

#

By def, x is a limit point of C when for all open $U \subset X$, s.t. $x \in U$, we have $(C \cap U) \setminus \{x\} \neq \phi$

for $x \in C$, suppose $\exists U_x$, $x \in U_x$, and $(C \cap U_x) \setminus \{x\} = \phi$

$\because (C \cap U_x) \setminus \{x\} = C \cap (U_x \setminus \{x\})$ and $C = G^c$

$\therefore U_x \setminus \{x\} \subseteq G$, and we have $U_x \cap E$ ~~untable~~ uncountable

i.e. G contains ~~exp~~ uncountable many points in E

But $G \cap E = \bigcup_{n \in A} (U_n \cap E)$ countable contradiction

$\therefore$ we know x is a limit point.

#

7.

$\mathbb{R}\setminus\{0\} = (-\infty, 0) \cup (0, \infty)$ union of open sets.

$\therefore \mathbb{R}\setminus\{0\}$ is open.

So $f^{-1}(\mathbb{R}\setminus\{0\}) = (f^{-1}(0))^c$ is open since f is continuous

Hence, $f^{-1}(0) = Z(f)$ is closed. #

8. if we can show $S = \{x \in X : f(x) \neq g(x)\}$ is open, we are done.

~~Suppose~~ fix $x \in S$, i.e. $f(x) \neq g(x)$.

Since Y is Hausdorff, there exist $U, V \subset Y$, s.t. $f(x) \in U$, $g(x) \in V$. $U \cap V = \phi$

$\therefore x \in f^{-1}(U) \cap g^{-1}(V)$, where $f^{-1}(U)$, $g^{-1}(V)$ both open.

so $f^{-1}(U) \cap g^{-1}(V)$ is also an open set.

Also, $\forall t \in f^{-1}(U) \cap g^{-1}(V)$, we have $f(t) \in U$, and $g(t) \in V$

Since $U \cap V \neq 0$, we have $f(t) \neq g(t) \Rightarrow t \in S$

$\therefore f^{-1}(U) \cap g^{-1}(V) \subseteq S$

Now, in sum, $\forall x \in S$, we can find an open set $f^{-1}(U) \cap g^{-1}(V) \subseteq S$.

$\therefore S$ is open. Therefore, $S^c = \{x \in X : f(x) = g(x)\}$ is closed.

if Q is a dense subset of X, i.e. $Q \subseteq X$, and $\overline{Q} = X$

if $Q \subseteq \{x \in X : f(x) = g(x)\}$, we have $X = \overline{Q} \subseteq \overline{\{x \in X : f(x) = g(x)\}} = \{x \in X : f(x) = g(x)\}$

since it's closed

$\Rightarrow X = \{x \in X : f(x) = g(x)\}$ #

7(a) $f: X \to \mathbb{R}$

$x \mapsto \rho(x,E) = \inf\{\rho(x,y) : y \in E\}$

f is u.c. if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. for all x, y with $\rho(x,y) < \delta$,

we have $|f(x) - f(y)| = |\rho(x,E) - \rho(y,E)| < \varepsilon$.

Now fix $\varepsilon > 0$, if $\rho(x,y) < \delta$

consider $|\rho(x,E) - \rho(y,E)| = |\inf\{\rho(x,z) : z \in E\} - \inf\{\rho(y,z) : z \in E\}|$

$\leq |\inf\{\rho(x,y) + \rho(y,z) : z \in E\} - \inf\{\rho(y,z) : z \in E\}|$

$\leq |\inf\{\delta + \rho(y,z) : z \in E\} - \inf\{\rho(y,z) : z \in E\}|$

$= |\delta + \inf\{\rho(y,z) : z \in E\} - \inf\{\rho(y,z) : z \in E\}|$

$= \delta$

Therefore, let $\delta = \varepsilon$, so for any $\varepsilon > 0$, if $\rho(x,y) < \varepsilon$ we will have $|\rho(x,E) - \rho(y,E)| < \varepsilon$. $\therefore$ f is uniformly continuous

Now want to show $\overline{E} = \{x : \rho(x,E) = 0\}$

By definition, $\{x \in X : \rho(x,E) = 0\} = \{x \in X : \inf\{\rho(x,y) : y \in E\} = 0\}$

if $x \in X$ satisfies $\inf\{\rho(x,y) : y \in E\} = 0$,

then $\forall n \in \mathbb{N}$, $\frac{1}{n} > 0$, there exist $y_n \in E$, s.t. $\rho(x, y_n) < \frac{1}{n}$.

$\therefore$ $\rho(x,E) = 0$ iff there is a sequence $\{y_n\}$ s.t. $\rho(x, y_n) < \frac{1}{n}$

i.e. $\rho(x, y_n) \to 0$

then $y_n \to x$. $\therefore$ $x \in \overline{E}$

Therefore, $\overline{E} = \{x \in X : \rho(x,E) = 0\}$ #

($\overline{E} \subset \{x \in X : \rho(x,E) = 0\}$ is easy to prove)

9(b) Let $G_n = \{x \in X : \rho(x, F) < \frac{1}{n}\}$.

From (a), we know $f: X \to \mathbb{R}$ defined by $f(x) = \rho(x, F)$ is uniformly continuous, so it's continuous. $\therefore G_n = \{x : \rho(x, F) < \frac{1}{n}\}$

$= f^{-1}(-\infty, \frac{1}{n})$ is open

$\therefore \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \{x \in X : \rho(x, F) < \frac{1}{n}\} = \{x \in X : \rho(x, F) = 0\}$

$\underset{\text{From (a)}}{=} \overline{F} \underset{\text{Since } F \text{ is closed.}}{=} F$

9(c)

$$f(x) = \frac{\rho(x, E)}{\rho(x, E) + \rho(x, F)}$$

easy to show $f(x)$ is continuous and $0 \le f \le 1$.

Now want to show $E = f^{-1}\{0\}$. $F = f^{-1}\{1\}$.

because E, F both are closed (since $\{0\}, \{1\}$ closed).

we have $E = \overline{E} = \{x \in X : \rho(x, E) = 0\}$

$F = \overline{F} = \{x \in X : \rho(x, F) = 0\}$

since $\rho(x, E) = 0$ iff $f(x) = 0$, $\rho(x, F) = 0$ iff $f(x) = 1$

Therefore. $E = f^{-1}\{0\}$. $F = f^{-1}\{1\}$

$\therefore E \subset f^{-1}(-1, \frac{1}{2})$. $F \subset f^{-1}(\frac{1}{2}, 2)$

i) $\forall x \in C$, since f continuous at x, then $\forall n \in \mathbb{N}$,

$f^{-1}(B_{\frac{1}{n}}(f(x)))$ is a neighborhood of x. ~~to~~ then there exist open $U_{x,\frac{1}{n}} \subseteq f^{-1}(B_{\frac{1}{n}}(f(x)))$, with $x \in U_{x,\frac{1}{n}}$.

Let $G_n = \bigcup_{x \in C} U_{x,\frac{1}{n}}$ open.

Claim: $C = \bigcap_{n=1}^{\infty} G_n$

First, notice that ~~$x \in G_n$~~. $\forall x \in C$, $x \in G_n$, for all n

$\therefore x \in \bigcap_{n=1}^{\infty} G_n \Rightarrow C \subseteq \bigcap_{n=1}^{\infty} G_n$

On the other hand, $\forall x_0 \in \bigcap_{n=1}^{\infty} G_n$, we have $x_0 \in \bigcup_{x \in C} U_{x,\frac{1}{n}}$. $\forall n$,

since $x_0 \in \bigcup_{x \in C} U_{x,\frac{1}{n}}$, exist $x' \in C$, s.t $x_0 \in U_{x',\frac{1}{n}}$

Then $U_{x',\frac{1}{n}}$ is a open set containing x_0, and

$\forall y \in U_{x',\frac{1}{n}}$, $|f(x_0) - f(y)| \leq |f(x_0) - f(x')| + |f(x') - f(y)| \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

Now we recall def. of continuity: f continuous at x, if $\forall n > 0$, exist an open set U containing x, s.t $\forall x' \in U$, $|f(x') - f(x)| < \frac{1}{n}$.

Now $\forall n > 0$, since $x_0 \in G_{2n}$, from the argument above, ~~we~~ we have there exist $U_{x',\frac{1}{2n}}$ containing x_0, s.t $f(U_{x',\frac{1}{2n}}) \subseteq B_{\frac{1}{n}}(f(x_0))$

Therefore, $x_0 \in C \Rightarrow \bigcap_{n=1}^{\infty} G_n \subseteq C$

$\therefore C = \bigcap_{n=1}^{\infty} G_n$ #

12.(a) if f is continuous w.r.t standard topology. then $f^{-1}(a,b)$ is open in X. for any $a, b \in \mathbb{R}$.

So in particular, $f^{-1}(-\infty, a) = \bigcup_{n=1}^{\infty} f^{-1}(a-n, a)$

$f^{-1}(a,\infty) = \bigcup_{n=1}^{\infty} f^{-1}(a, a+n)$ both open

$\therefore$ f is continuous w.r.t T_u. T_l

$\therefore$ f is u.s.c and l.s.c

Conversely, if f is u.s.c and l.s.c. then $f^{-1}(-\infty, b)$ is open in X for all $b \in \mathbb{R}$ and $f^{-1}(a,\infty)$ is open in X for all $a \in \mathbb{R}$,

so that $f^{-1}(a,b) = f^{-1}(-\infty, b) \cap f^{-1}(a,\infty)$ is open in X

$\therefore$ f is continuous

(b) for all $z \in \mathbb{R}$,

$(f+g)^{-1}(-\infty, z) = \{x \in X : (f+g)(x) < z\} = \{x \in X : f(x) < a, g(x) < b, a+b < z\}$

$$= \bigcup_{\substack{a,b \in \mathbb{R} \\ a+b<z}} \underbrace{f^{-1}(-\infty, a) \cap g^{-1}(-\infty, b)}_{\Downarrow \text{ open since } f.\ g \text{ u.s.c}}$$

$\therefore$ $(f+g)^{-1}(-\infty, z)$ is open for all z

$\therefore$ $f+g$ is u.s.c

Also, $(\lambda f)^{-1}(-\infty, z) = \{x \in X : f(x) < \frac{z}{\lambda}\} = f^{-1}(-\infty, \frac{z}{\lambda})$ open

$\therefore$ λf is u.s.c

(c). Suppose $\mathcal{F} = \{f : X \to \mathbb{R}\}$ collection of u.s.c functions

$g(x) = \inf\{f(x) . f \in \mathcal{F}\}$

<u>claim</u>. $g^{-1}(-\infty, a) = \bigcup_{f \in \mathcal{F}} f^{-1}(-\infty, a)$

if $x \in g^{-1}(-\infty, a)$, $\Rightarrow g(x) < a$, i.e $\inf\{f(x): f \in \mathcal{F}\} < a$

$\therefore$ exist $f_0 \in \mathcal{F}$, s.t $f_0(x) < a$. $\therefore x \in f_0^{-1}(-\infty, a)$

$\therefore \quad x \in \bigcup_{f \in \mathcal{F}} f^{-1}(-\infty, a) \quad \Rightarrow \quad g^{-1}(-\infty, a) \subseteq \bigcup_{f \in \mathcal{F}} f^{-1}(-\infty, a)$

On the other hand, if $x \in \bigcup_{f \in \mathcal{F}} f^{-1}(-\infty, a)$, exist $f_0 \in \mathcal{F}$,

$x \in f_0^{-1}(-\infty, a)$, then $f_0(x) < a$, $\therefore g(x) = \inf_{f \in \mathcal{F}} f(x) \le f_0(x) < a$

$\therefore x \in g^{-1}(-\infty, a) \quad \Rightarrow \quad \bigcup_{f \in \mathcal{F}} f^{-1}(-\infty, a) \subseteq g^{-1}(-\infty, a)$

$\therefore g^{-1}(-\infty, a) = \bigcup_{f \in \mathcal{F}} f^{-1}(-\infty, a)$ open (since f u.s.c)

$\therefore g$ is u.s.c

(d)

Notice $(-f)^{-1}(-\infty, a) = f^{-1}(-a, \infty)$

$f^{-1}(a, \infty) = (-f)^{-1}(-\infty, -a)$

easy to show