HW4 Solution

13. "$\Rightarrow$" Suppose that X is not compact. Then there exist an open cover $\{U_\alpha\}_{\alpha\in A}$ say, of X, with property that $X\setminus\bigcup_{i=1}^{n} U_{\alpha_i} \neq \phi$ for any finite selection $\alpha_1, \dots, \alpha_n \in A$.

Setting $F_\alpha = X\setminus U_\alpha$, we thus have a collection $\mathcal{F} := \{F_\alpha\}_{\alpha\in A}$ of closed sets in X with property.

$$\bigcap_{i=1}^{n} F_{\alpha_i} = \bigcap_{i=1}^{n} (X\setminus U_{\alpha_i}) = X\setminus \bigcup_{i=1}^{n} U_{\alpha_i} \neq \phi \quad \text{for any finite } \alpha_1, \dots, \alpha_n \in A$$

That is, $\mathcal{F}$ has the finite intersection property, then we have $\bigcap_{\alpha\in A} F_\alpha \neq \phi$. i.e. $X\setminus\bigcup_{\alpha\in A} U_\alpha \neq \phi$

which contradicts the assumption that U is open cover of X

$\therefore$ X is compact.

"$\Leftarrow$" Assume X is compact. and suppose (for contradiction) that there is a collection $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$ of closed subsets of X, with the f.i.p, but $\bigcap_{\alpha\in A} F_\alpha = \phi$.

set $U_\alpha = X\setminus F_\alpha$, s.t. $\bigcup_{\alpha\in A} U_\alpha = X\setminus \bigcap_{\alpha\in A} F_\alpha = X$

i.e. $\{U_\alpha\}$ is an open cover of X. By compactness, there is a finite subcover. $X = \bigcup_{i=1}^{n} U_{\alpha_i}$,

where $\phi = X\setminus\left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) = \bigcap_{i=1}^{n} F_{\alpha_i} \rightarrow$ contradiction with f.i.p

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6. $\Rightarrow$ suppose that f is continuous,

All open sets in $G(f)$ are of the form $\{(x, f(x)) \mid x \in U, f(x) \in V\}$ for U open in X, V open in $\mathbb{R}$.

This set is equivalent to $\{(x, f(x)) \mid x \in U \cap f^{-1}(V)\}$

Now suppose $C = \{O_\alpha \mid O_\alpha$ open in $G(f)\}$ is an open cover of $G(f)$, since X is compact and all open sets in $G(f)$ depend only on elements of X, i.e

$C = \bigcup_\alpha O_\alpha = \bigcup_\alpha \{(x, f(x)) \mid x \in U_\alpha \cap f^{-1}(V_\alpha)\}$, since $\{U_\alpha \cap f^{-1}(V_\alpha)\}$ is an open cover for X, and X is compact, then we can find a finite subcover, then we can get a finite subcover of C correspondingly.

$\therefore$ $G(f)$ is compact.

$\Leftarrow$ Now suppose $G(f)$ compact.

Let $E \subset \mathbb{R}$ closed, the projection mapping $\pi_2 : G(f) \to \mathbb{R}$ is continuous, so $\pi_2^{-1}(E)$ is closed in $G(f)$, we have that $\pi_2^{-1}(E)$ is a closed subset of a compact set, so it's compact.

We also have $\pi_1 : G(f) \to X$ continuous, $\therefore$ $\pi_1(\pi_2^{-1}(E)) = f^{-1}(E)$ is compact in X. and X is Hausdorff so $f^{-1}(E)$ is closed, $\therefore$ f is continuous

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1. "injective": if $f(x)=f(y)$, then $\rho(f(x), f(y))=0=\rho(x,y)$

$\Rightarrow x=y$

"surjective". Let $Y=f(X)$, if $Y\neq X$. i.e. $X\setminus Y\neq\phi$,

let $x_0\in X\setminus Y$, $\delta=\rho(x_0, Y)$.

① $\delta=0$. then $x_0\in\overline{Y}$. However, f is continuous so $f(X)$ is compact, But X is a metric space, so it's Hausdorff

then all compact subsets are closed.

$\therefore Y$ is closed. $Y=\overline{Y}$. $\Rightarrow x_0\in Y$ contradiction.

② $\delta>0$

define sequence $\{x_n\}$ by $x_{n+1}=f(x_n)$.

Then for all $m<n$,

$\rho(x_n, x_m) = \rho(f(x_{n-1}), f(x_{m-1}))$

$= \rho(x_{n-1}, x_{m-1})$

$\vdots$

$= \rho(x_{n-m}, x_0) \geq \delta$

$\therefore X$ is not sequentially compact.

$\Rightarrow X$ is not compact. ⊗

$\therefore f$ is bijective

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18. Let X be a compact space, and

$f: X \to \mathbb{R} \cup \{-\infty\}$ is an u.s.c function.

Let $A = f(X)$ and $\forall c \in A$, define

$F_c = \{x \in X : f(x) \geq c\} = f^{-1}((-\infty, c)^c)$

Since $^c(-\infty, c)$ closed in $\mathcal{T}_u$ and f continuous,

$\therefore F_c$ is closed, also, notice that $F_c \neq \phi$.

Now consider any finite subcollection $\{F_{c_1}, \cdots, F_{c_n}\}$

$\subseteq \{F_c : c \in A\}$, $\bigcap_{i=1}^{n} F_{c_i} = \{x \in X : f(x) \geq \max\{c_1, \cdots, c_n\}\}$,

That is, $\bigcap_{i=1}^{n} F_{c_i} = F_{c_j} \neq \phi$, where $c_j = \max\{c_1, \cdots, c_n\}$

Therefore, T has finite intersection property.

By problem 13, Since X is compact, we have

$\bigcap_{c \in A} F_c \neq \phi$, so exist $y \in \bigcap_{c \in A} F_c$, such that

$f(y) \geq c$ for all $c \in A$.

$\therefore$ f assumes maximum value at y.

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