HW5 Solution.

1. <u>Proof</u>. need to show for E closed under F, $\Rightarrow$ E closed under T.

Since all closed subset of compact space are compact, so E is compact under F.

$\because$ $T \subset F$, any open cover of E under T is also an open cover of E under F.

E is compact under F, so we can find a finite subcover in F, then it's also a finite subcover in T. (Since we first suppose it's open cover under T)

$\therefore$ E is compact under T.

$\because$ T is Hausdorff.

$\therefore$ E is closed under T

$\therefore$ $F \subset T$

Therefore, $F = T$

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2. Let $D = \{(x, x) : x \in X\}$.

only need to show $D^c = \{(x, y) : x \neq y\}$ is open.

for $(x, y) \in D^c$, $x \neq y$,

then since X Hausdroff, exist disjoint open sets

U, V, s.t $x \in U$, $y \in V$, $U \cap V = \phi$

then we have $U \times V \subset D^c$ and $U \times V$ is open in $X \times X$

$\therefore D^c$ is open

Then we know D is closed. #

3. Let $C \subseteq G$ be a component of G.

if $x \in C$, since G is locally connected, there exists

an open connected neighborhood, U_x of x.

Now since U_x is connected and $x \in U_x$, $\therefore U_x \subseteq C$.

$\Rightarrow C = \bigcup_{x \in C} x = \bigcup_{x \in C} U_x$ union of open sets

$\therefore C$ is open

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22. Let U be open subset of $\mathbb{R}$.

Since $\mathbb{R}$ is connected, then it's also locally connected.

$\therefore$ From 21, we know every component of U is open.

and $C_x \cap C_y = \phi$ or $C_x = C_y$.

$\therefore$ $U = \bigcup_x C_x$ must be union of disjoint component.

also we know every component is an open interval (since it's open)

$\therefore$ U is the union of disjoint sequence of open interval

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23. (i) Let F be a set of convex sets in $\mathbb{R}^d$, $D = \bigcap_{C \in F} C$

$\forall x, y \in D$, $x, y \in C$ for all $C \in F$

and since each F convex, $\therefore$ $tx + (1-t)y \in C$ $0 \le t \le 1$, for all $C \in F$

$\therefore$ $tx + (1-t)y \in D$ $0 \le t \le 1$

$\therefore$ D is convex

(ii) Now let $C \subset \mathbb{R}^d$ convex. For $x, y \in C$.

define $\gamma_{xy} : [0,1] \to C$ as

$\gamma_{xy}(t) = ty + (1-t)x \in C$ $\forall t$ (since C convex)

$\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$

then it's easy to see that C is connected.

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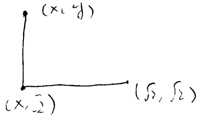
24. $X = \{(x,y) : \text{either } x \text{ or } y \text{ irrational})\}$

First, show that $\forall (x,y) \in X$, there is a continuous path to $(\sqrt{2}, \sqrt{2}) \in X$.

Since $(x,y) \in X$, w.l.o.g., suppose $x \in \mathbb{Q}^c$



construct a path as in graph.

(i) $(x,y) \to (x,\sqrt{2})$.

$\gamma(t) = (x, t\sqrt{2} + (1-t)y)) \in X$

(ii) $(x,\sqrt{2}) \to (\sqrt{2},\sqrt{2})$

$\alpha(t) = (t\sqrt{2} + (1-t)x, \sqrt{2}) \in X$

$\therefore \forall (x_1, y_1), (x_2, y_2) \in X$.

can construct a continuous path via $(\sqrt{2},\sqrt{2})$.

$(x_1, y_1) \to (\sqrt{2},\sqrt{2}) \to (x_2, y_2)$

$\therefore$ X is connected

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