HW (6) Solution

1. $\mu(E\cup F\cup G) = \mu(E) + \mu(F) + \mu(G) - \mu(E\cap F) - \mu(E\cap G) - \mu(F\cap G)$
$+\mu(E\cap F\cap G)$

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n}\mu(E_i) - \sum_{1\le i<j\le n}\mu(E_i\cap E_j) + \dots + (-1)^n\mu\left(\bigcap_{i=1}^{\infty} E_i\right)$$

2. Let $F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$

can check that F_i pairwise disjoint, and $F_n \subseteq E_n$

and you could show

$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\mu(F_n) \le \mu(E_n)$.

$$\therefore \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right)$$
$$= \sum_{n=1}^{\infty}\mu(F_n)$$
$$\le \sum_{n=1}^{\infty}\mu(E_n)$$

3. let $E = \bigcap_{i=1}^{\infty} E_i$, $F_i = E_i \setminus E_{i+1}$.

Then $E_1 \setminus E = \bigcup_{i=1}^{\infty} F_i$

and $F_i \cap F_j = \phi$ for $i \neq j$. Hence,

$$\mu(E_1 \setminus E) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i)$$

$$= \sum_{i=1}^{\infty} \mu(E_i \setminus E_{i+1})$$

But $\mu(E_1) = \mu(E) + \mu(E_1 \setminus E)$,

$\mu(E_i) = \mu(E_{i+1}) + \mu(E_i \setminus E_{i+1})$

Since $\mu(E_i) \le \mu(E_1) < \infty$, $\therefore$ we have

$\mu(E_1 \setminus E) = \mu(E_1) - \mu(E)$ and $\mu(F_i) = \mu(E_i \setminus E_{i+1}) = \mu(E_i) - \mu(E_{i+1})$

$$\therefore \mu(E_1) - \mu(E) = \sum_{i=1}^{\infty} (\mu(E_i) - \mu(E_{i+1}))$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} (\mu(E_i) - \mu(E_{i+1})) \right)$$

$$= \lim_{n\to\infty} (\mu(E_1) - \mu(E_n))$$

$$= \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$$

$$\therefore \mu(E) = \lim_{n\to\infty} \mu(E_n)$$

if $\mu(E_1) = \infty$, consider

$E_n = [n, \infty)$, then $\mu(E_n) = \infty$, $\forall n$, $\therefore \lim_{n\to\infty} \mu(E_n) = \infty$

However, $\bigcap_{n=1}^{\infty} E_n = \phi$.

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4. $A_n = \bigcap_{m=n}^{\infty} E_m$

Then, $A_n \subset A_{n+1}$, $\forall n$.

$\therefore \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$ $\quad \because \mu(A_n) = \mu(\bigcap_{m=n}^{\infty} E_m) \le \inf_{m\geq n} \mu(E_m)$

$$\therefore \mu(\liminf E_n) = \mu(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m)$$

$$= \mu(\bigcup_{n=1}^{\infty} A_n) = [illegible]$$

$$= \lim_{n\to\infty} (A_n)$$

$$\le \lim_{n\to\infty} \inf_{m\geq n} \mu(E_m) = \liminf \mu(E_n)$$

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5. we know $\mu(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \mu(E_n) < \infty$

and for all n,

$$\bigcup_{k=n}^{\infty} E_k = E_n \cup \bigcup_{k=n+1}^{\infty} E_k \subseteq \bigcup_{k=n+1}^{\infty} E_k$$

$$\therefore \mu(\limsup E_n) = \lim_{n\to\infty} \mu(\bigcup_{m=n}^{\infty} E_m)$$

$$\le \lim_{n\to\infty} (\sum_{m=n}^{\infty} \mu(E_m))$$

Since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. $\therefore$ $\sum_{m=n}^{\infty} \mu(E_m) \xrightarrow{n\to\infty} 0$

$\therefore \mu(\limsup E_n) = 0$

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10. Let $A = \{E \subset Y : f^{-1}(E) \in \mathcal{B}(X)\}$

<u>Claim</u> A is σ-algebra

easy to check the properties of σ-algebra

Then since f is continuous, if $E \subset Y$ closed, $f^{-1}(E) \subset X$

also closed. So $E \in A$

$\therefore$ A contains all closed sets of Y

$\Rightarrow$ A is σ-algebra generated by closed sets of Y

$\Rightarrow$ A is Borel algebra .

$\therefore$

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