HW 6.

1. Let $E = \{x_1, x_2, x_3, \dots\}$

$\forall \varepsilon > 0$, define $I_n = (x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}})$, where $x_n \in I_n$

$\therefore E \subseteq \bigcup_{n=1}^{\infty} I_n$

$\therefore m^*(E) \le m^*(\bigcup_{n=1}^{\infty} I_n) \le \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

Since ε arbitrary, we have $m^*(E) = 0$.

Now need to consider the measurability issue:

Let $A \subseteq \mathbb{R}$,

we have $m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c)$

Since $m^*(E) = 0$, $A \cap E \subseteq E$, we have $m^*(A \cap E) = 0$

Similarly, $m^*(A \cap E^c) \le m^*(A)$

$\therefore m^*(A \cap E^c) + m^*(A \cap E) \le m^*(A)$

Therefore, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$\therefore E$ is measurable, and $m(E) = 0$

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2. (a) Let $E \subseteq \mathbb{R}^n$ closed.

$d(x, E) = \inf \{ \ell(x, y) : y \in E \}$.

We know $\{x : d(x, E) = 0\} = \bar{E} = E$ (Since E closed)

consider $U_n = \{ x : d(x, E) < \frac{1}{n} \}$

easy to show that U_n is open and

$\bigcap_{n=1}^{\infty} U_n = \{x : d(x, E) = 0\} = E$ $\therefore E$ is G_δ.

(b) consider $E = \mathbb{Q}$ in $\mathbb{R}$.

$E = \bigcup_{n=1}^{\infty} \{q_n\}$ $\{q_n\}$ closed. $\therefore$ $\cancel{D}$ E is an F_δ set

Suppose E is also a G_δ set, i.e. $E = \bigcap_{n=1}^{\infty} U_n$ for U_n open.

Since $E = \bigcap_{n=1}^{\infty} U_n$ dense, can show that U_n is dense, $\forall n$.

$\therefore$ ~~U_n~~ $U_n \setminus \{q_n\}$ also a dense set.

Notice that $U_n \setminus \{q_n\} = U_n \cap \{q_n\}^c$ open.

By Blair Category theorem, $\bigcap_{n=1}^{\infty} U_n \setminus \{q_n\}$ dense.

i.e. exist $s \in \bigcap_{n=1}^{\infty} U_n \setminus \{q_n\}$, But $\bigcap_{n=1}^{\infty} U_n \setminus \{q_n\} \subseteq \bigcap_{n=1}^{\infty} U_n = E$.

$\therefore s \in \cancel{D}^E$ contradiction since $q_n \notin \bigcap_{n=1}^{\infty} U_n \setminus \{q_n\}$, $\forall n$

$\therefore$ $\cancel{D}$ E is not a G_δ.

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3. Let D be an open disk, if

$D = \bigcup_{n=1}^{\infty} R_n$ when R_n disjoint open rectangles.

~~So~~ D is connected, However, R.H.S not connected

contradiction

$\otimes$

4. ① if E is measurable under usual def.

$\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \lambda(C_n) \mid \bigcup C_n \supset E\}$

$\forall \varepsilon > 0$, $\exists$ open U containing E, s.t $\mu^*(U) \le \mu^*(E) + \varepsilon$

Hence, since E measurable, E^c measurable,

$\therefore \forall \varepsilon > 0$, $\exists U \subseteq E^c$, s.t $\mu^*(U) < \mu^*(E^c) + \varepsilon$

i.e. $\mu^*(U) - \mu^*(E^c) < \varepsilon$.

Then, $\mu^*(U \setminus E^c) = \mu^*(U) - \mu^*(U \cap E^c)$

$= \mu^*(U) - \mu^*(E^c) < \varepsilon$

$\Rightarrow \mu^*(U \setminus E^c) < \varepsilon$

$U \setminus E^c = U \cap (E^c)^c = U \cap E = E \cap (U^c)^c = E \setminus U^c$

$\therefore \mu^*(E \setminus U^c) < \varepsilon$ where $U^c \subset E$ closed

$\therefore \exists F$ closed in E, s.t $\mu^*(E \setminus F) < \varepsilon$

② if $\forall k \in \mathbb{N}$, $\exists F_k$, s.t $F_k \subset E$. $\mu^*(E \setminus F_k) < \frac{1}{k}$

Let $B = \bigcup_{k=1}^{\infty} F_k$ $\Rightarrow$ B measurable.

and $E \setminus B \subset E \setminus F_k$ $\forall k$.

$\therefore \mu^*(E \setminus B) \le \mu^*(E \setminus F_k) < \frac{1}{k}$ $\forall k$

$\therefore \mu^*(E \setminus B) = 0$. $\therefore E \setminus B$ measurable.

$\Rightarrow B$, $E \setminus B$ measurable

$\therefore E = B \cup (E \setminus B)$ also measurable.

$\square$

5. $A \subset E \subset B$. $m(A) = m(B)$

$\therefore m^*(A) \le m^*(E) \le m^*(B) \Rightarrow m^*(E) = m^*(A) = m^*(B)$

~~$\therefore m^*(E) = m^*(A) + m^*(E \setminus A)$ since $E \setminus A$, A disjoint~~

$\therefore m^*(E \setminus A) \le m^*(B \setminus A) = 0 \Rightarrow m^*(E \setminus A) = 0$

$\therefore E \setminus A$ measurable

$\therefore E = A \cup (E \setminus A)$ also measurable #

6.(a) Let $E \subset N$ measurable and let $\{q_n\}_{n=1}^{\infty} = \mathbb{Q} \cap [0,1)$

let $x \oplus y$ denote addition mod 1, Now consider

$$\bigcup_{n=1}^{\infty} E \oplus q_n.$$

Since $E \subset N$ and $\bigcup N \oplus q_n = [0,1)$, we have that $\bigcup E \oplus q_n \subseteq [0,1)$.

Therefore, we see that

$$m\left(\bigcup_{n=1}^{\infty} E \oplus q_n\right) \le m([0,1)) = 1$$

However, E is measurable and the Lebesgue measure is translation invariant, so $E \oplus q_n$ must also be Lebesgue measurable, with $m(E \oplus q_n) = m(E)$.

This gives $1 \ge m\left(\bigcup_{n=1}^{\infty} E \oplus q_n\right) = \sum_{n=1}^{\infty} m(E \oplus q_n) = \sum_{n=1}^{\infty} m(E)$

If $m(E) > 0$, the $\sum m(E) = \infty$ contradiction

$\therefore m(E) = 0$

(b) Mimic the construction of N.

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7.1. Let $U \subseteq \mathbb{R}$ open.

g continuous $\therefore g^{-1}(U)$ also open.

and f measurable, $\therefore f^{-1}(g^{-1}(U))$ also measurable

and $(f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$

$\therefore g\circ f$ measurable function.

7.2. $\forall x \in [0,1]$, it has binary representation

(a)

$$\therefore f_0\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{\frac{a_n}{2}}{2^n} \quad \text{where } \frac{a_n}{2} \in \{0,1\}$$

$\therefore$ easy to see that f_0 is surjective.

(b) define f by

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in C \\ \text{later.} & \text{if } x \notin C \end{cases}$$

Now, for $x \in [0,1]\setminus C$, there exist $y_1, y_2 \in C$ s.t. y_1 is the largest elt in C less than x and y_2 is the smallest elt in C greater than x, then let

$x = \sum x_n 3^{-n}$ for some k. $x_k = 1$

$$y_1 = \sum_{n=1}^{k-1} \frac{x_n}{3^n} + \sum_{n=k+1}^{\infty} \frac{2}{3^n}$$

$$y_2 = \sum_{n=1}^{k-1} \frac{x_n}{3^n} + \frac{2}{3^k}$$

(you could check y_1, y_2 satisfies above statement)

also, can check $f_0(y_1) = f_0(y_2)$

Then
$$f(x) = \begin{cases} f_0(x) & \text{if } x \in C \\ f_0(y_1) = f_0(y_2) & \text{if } x \notin C \end{cases}$$

$\dagger$

3. Let B be non-measurable set in $[0,1]$

Consider $f^{-1}(B) \cap C = D$ (C cantor set)

(1) D is measurable

$f^{-1}(B) \cap C \subset C$

$\therefore m^*(f^{-1}(B) \cap C) \leq m^*(C) = 0$

$\therefore m^*(f^{-1}(B) \cap C) = 0$

$\therefore D$ is measurable

(2) $f(D)$ is non-measurable

Since f_0 is surjective, non-decreasing.

$f(D) = f_0(D) = B$, non-decreasing

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4. (a) $F(x) = \frac{1}{2}(x + f(x))$

continuous

both x, $f(x)$ non-decreasing, $\nearrow$ $\Rightarrow$ $F(x)$ non-decreasing, continuous

since for $x < y$,

$F(x) < F(y)$ strictly

$\therefore F$ is continuous and 1-1.

$\therefore F^{-1}$ also continuous

$\therefore G$ is continuous

(b) $B = G^{-1}(C) = (F^{-1})^{-1}(C) = F(C)$

Since F continuous, ~~closed~~ C Borel set

$\therefore B = F(C)$ also Borel set

$$m(F([0,1]\setminus C)) = m(F(\bigcup_{n=1}^{\infty}(a_n, b_n)))$$

$$= \sum_{n=1}^{\infty} m(F(a_n, b_n))$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} m(a_n, b_n))$$

$$= \frac{1}{2} m(\bigcup_{n=1}^{\infty}(a_n, b_n))$$

$$= \frac{1}{2} m([0,1]\setminus C))$$

$$= \frac{1}{2}$$

$F(C)$ and $F([0,1]\setminus C)$ disjoint, $\therefore m(F(C)) + m(F([0,1]\setminus C)) = m(F([0,1]))$

$\therefore m(F(C)) = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore m(B) = \frac{1}{2}$

(c) since $m(B) > 0$, B contains non-measurable set N

$\Rightarrow N \subset B = F(C)$

here, F is 1-1, so $F^{-1}(N)$ exists and $F^{-1}(N) \subset C$

$\Rightarrow m^*(F^{-1}(N)) \le m^*(C) = 0$

$\therefore m^*(F^{-1}(N)) = 0 \Rightarrow F^{-1}(N)$ measurable

So $E = F^{-1}(N) \rightarrow G^{-1}(E) = F(E) = N$ is not measurable

even if E is measurable

But then E is an example of measurable, but not Borel set.
if E is Borel set, $F(E) = N$ is Borel set, But N is not Borel set (Borel set is measurable).

Now, let $g = \mathbb{1}_E$. g is measurable since E is measurable

But $g \circ G$ is not measurable. since

$\{x \in [0,1] \mid g \circ G(x) > \frac{1}{2}\}$

$= \{x \in [0,1] \mid G(x) \in E\} = G^{-1}(E) = F(E) = N$ not measurable

$\therefore g \circ G$ not measurable

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