HW8

6. Suppose not, then there exists $\varepsilon > 0$, s.t.

$\forall n$, exist A_n with $\mu(A_n) < \frac{1}{2^n}$, and $\nu(A_n) \geq \varepsilon$

Let $A = \limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$

since $\sum_{n=1}^{\infty} \mu(A_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$

by problem 5, chap 9, we have $\mu(A) = 0$

On the other hand, $\nu(A) = \cancel{\lim_{n\to\infty} \nu(\bigcap_{m=n}^{\infty} A_m)} = \lim_{n\to\infty} \nu(\bigcup_{m=n}^{\infty} A_m)$

Since $\bigcup_{m=n}^{\infty} A_m \supseteq A_n$

$\therefore \nu(\bigcup_{m=n}^{\infty} A_m) \geq \nu(A_n) \geq \varepsilon$.

$\therefore \nu(A) \geq \varepsilon$ which contradicts with $\nu(A) = \int_A f \, d\mu = 0$

since $\mu(A) = 0$.

7. (a) $F(t) = \mu\{x : |f| > t\} = \mu\{x : |f|^p > t^p\}$

by Chebyshev's inequality,

$F(t) = \mu\{x : |f|^p > t^p\} \leq \frac{1}{t^p} \int |f|^p \, d\mu = C t^{-p}$

where $C = \int |f|^p \, d\mu < \infty$

(b) Suppose $\sum_{n=1}^{\infty} F(n) < \infty$, we then have

$\sum_{n=1}^{\infty} F(n) = \sum_{n=1}^{\infty} n \mu\{x : n \leq |f(x)| \leq n+1\}$

$\int |f| \, d\mu \overset{?}{=} \sum_{n=0}^{\infty} \int_{\{n \leq |f(x)| \leq n+1\}} |f| \, d\mu \leq \sum_{n=0}^{\infty} (n+1) \mu\{x : n \leq |f| \leq n+1\} = \mu(X) + \sum_{n=1}^{\infty} F(n) < \infty$

Conversely, if $\int |f| < \infty$

Since $\int |f| d\mu \geq \sum n \mu(x: n \leq |f| \leq n+1)$

$\therefore \sum F(n) < \int |f| d\mu < \infty$

(c) $\mu\{x: |f(x)|^r > t\} = \mu\{x: |f| > t^{\frac{1}{r}}\} = F(t^{\frac{1}{r}}) \leq \frac{c}{t^{\frac{p}{r}}}$

Then $\int |f(x)|^r d\mu < +\infty \Leftrightarrow \sum_{n=1}^{\infty} \mu\{x: |f|^r > t\} < \infty$

consider $\sum_{n=1}^{\infty} \mu\{x: |f|^r > t\} \leq \sum_{n=1}^{\infty} \frac{c}{n^{\frac{p}{r}}} < \infty$ since $\frac{p}{r} > 1$

$\therefore \int |f|^r d\mu < \infty$ for all $0 < r < p$.

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8. From 7(b) we have

$$t F(t) \leq \sum_{n=t}^{\infty} n \mu(\{x: n-1 \leq |f(x)| \leq n\}) < \infty$$

let $t \to \infty$

$$\lim_{t \to \infty} t F(t) \leq \lim_{t \to \infty} \sum_{n=t}^{\infty} n \mu(\{x: n-1 \leq |f| \leq n\}) = 0$$

9. Given $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$, we have

$$\sum_{n=1}^{\infty} \int |f_n| d\mu = \int \sum_{n=1}^{\infty} |f_n| d\mu < \infty$$

Therefore. we have $\sum_{n=1}^{\infty} |f_n| < \infty$ a.e

Now, $\left|\sum_{n=1}^{\infty} f_n\right| \leq \sum_{n=1}^{\infty} |f_n| < \infty$ a.e

So $\int \sum_{n=1}^{\infty} f_n \, d\mu \leq \int \left|\sum_{n=1}^{\infty} f_n\right| d\mu \leq \int \sum_{n=1}^{\infty} |f_n| \, d\mu < \infty$

$\therefore \sum_{n=1}^{\infty} f_n$ converge to a summable function a.e.

Now, let $g = \sum_{n=1}^{\infty} |f_n|$. Then. $\left|\sum_{n=1}^{N} f_n\right| \leq g$ a.e

By D.C.T,

$$\int \sum_{n=1}^{\infty} f_n d\mu = \int \lim_{N\to\infty} \sum_{n=1}^{N} f_n d\mu$$

$$= \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n d\mu$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n d\mu$$

$$= \sum_{n=1}^{\infty} \int f_n d\mu$$

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Q:

1. (a) $F(x) = \int_0^x f\,dm$ for $x \geq 0$, $F(x) = -\int_x^0 f\,dm$ for $x < 0$

for $x \geq 0$, $|F(x+\delta) - F(x)| = |\int_{[x,x+\delta]} f\,dm|$

since f integrable, and $m([x,x+\delta]) = \delta$.

Therefore, $\forall \varepsilon > 0$, $\exists \delta'$, s.t. $\delta < \delta'$, then $|F(x+\delta) - F(x)| < \varepsilon$

$\therefore F$ is continuous on $\mathbb{R}$. (Similar for $x < 0$)

(b) Let $E = \{x : f(x) > 0\}$, $E_n = \{x : f(x) \geq \frac{1}{n}\}$

Then $E = \bigcup_{n=1}^{\infty} E_n$

consider when $x \geq 0$, let $g_n = \frac{1}{n} \mathbb{1}_{E_n \cap [0,x]}$.

we have $0 \leq g_n \leq f$, and $\frac{1}{n}\mu(E_n \cap [0,x]) = \int g_n\,d\mu \leq \int \mathbb{1}_{[0,x]} f = F(x) = 0$

$\therefore \mu(E_n \cap [0,x]) = 0$ for all n.

By the same argument, we have $\mu(E_n \cap [x,0]) = 0$ for $x < 0$

$\therefore \underset{x<0}{\mu(E \cap [x,0])} = \underset{x>0}{\mu(E \cap [0,x])} = 0$

$\therefore \mu(E) = 0$

$\therefore f = 0$ a.e

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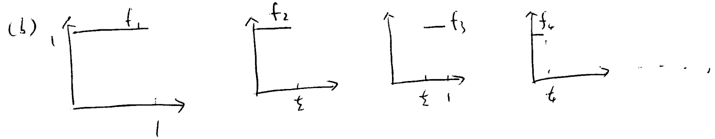
11. (a) $\forall \varepsilon > 0, \quad \{|f_n - f| > \varepsilon\} \subset \bigcup_{i=n}^{\infty} \{|f_i - f| > \varepsilon\}$

Since $f_n \to f$ a.e i.e. $\mu\left(\bigcup_{i=n}^{\infty} \{|f_i - f| > \varepsilon\}\right) \xrightarrow{n\to\infty} 0$. $\forall \varepsilon$

$\therefore \mu\{|f_n - f| > \varepsilon\} \xrightarrow{n\to\infty} 0 \quad \forall \varepsilon > 0$

$\therefore f_n \to f$ in measure

(b)



"Moving Function"

$f_n \not\to 0$ a.e, However, $f_n \to 0$ in measure

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12. Let $A = \{x : f_n(x) \not\to 0\}$

Define $A_{n,m} = \{x : |f_n(x)| > \frac{1}{m}\}$

$\therefore A = \bigcup_{m=1}^{\infty} \{x \in A_{n,m}$ for infinitely many $n\}$

$= \bigcup_{m=1}^{\infty} \underbrace{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_{k,m}}_{B_m}$

$\therefore \mu(A) = \mu\left(\bigcup_{m=1}^{\infty} B_m\right)$

Now, $\mu(B_m) \le \mu\left(\bigcup_{k=n}^{\infty} A_{k,m}\right) \le \sum_{k=n}^{\infty} \mu(A_{k,m})$

$= \sum_{k=n}^{\infty} \mu(\{x : |f_n(x)| > \frac{1}{m}\}) \to 0$

$\therefore \mu(B_m) = 0. \quad \forall m$

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$\therefore \mu(A) = 0$

17.

$\Rightarrow$ ~~A~~ If $\int |f_n - f| \, d\mu \to 0$

~~Since~~ $|f_n| - |f| \le |f_n - f|$

$\therefore \int |f_n| \, d\mu - \int |f| \, d\mu \le \int |f_n - f| \, d\mu.$

Since $\int |f_n - f| \, d\mu \to 0$, then $\forall \varepsilon > 0$, $\exists N$, s.t. $\left|\int |f_n - f| \, d\mu\right| < \varepsilon$ for all $n > N$

Therefore $\left|\int |f_n| \, d\mu - \int |f| \, d\mu\right| < \varepsilon$ for all $n > N$ as well

$\therefore \int |f_n| \, d\mu \to \int |f| \, d\mu$

$\Leftarrow$ Assume $\int |f_n| \, d\mu \to \int |f| \, d\mu$

Then there exist N, s.t. $n > N$, $\left|\int |f_n| \, d\mu - \int |f| \, d\mu\right| < \int |f| \, d\mu$

That is $\int |f_n| \, d\mu < 2 \int |f| \, d\mu$ for all $n > N$

Consider $|f_n - f| \le |f_n| + |f|$

Thus $\int |f_n - f| \, d\mu \le \int |f_n| \, d\mu + \int |f| \, d\mu \le 3 \int |f| \, d\mu$

By D.C.T,

$$\lim_{n \to \infty} \int |f_n - f| \, d\mu = \int \underbrace{\lim |f_n - f|}_{= 0} \, d\mu = 0$$

$\boxplus$

18. Notice that $\{x : \sqrt{f(x)} > t\} = \{x : f(x) > t^2\} \in A$.

Similar for g.

By thm 6.29,

$$\left(\int f\,d\mu\right)^{\frac{1}{2}} \left(\int g\,d\mu\right)^{\frac{1}{2}} \geq \left|\int \sqrt{fg}\,d\mu\right|$$

Since $fg \geq 1$, $\sqrt{fg} \geq 1$.

Thus, $\left|\int \sqrt{fg}\,d\mu\right| \geq \left|\int 1\,d\mu\right| = 1$

Hence $\left(\int f\,d\mu\right)\left(\int g\,d\mu\right)^{\frac{1}{2}} \geq 1$

$\therefore \left(\int f\,d\mu\right)\cdot\left(\int g\,d\mu\right) \geq 1$ #

19. $0 < r < s$, $\int |f|^r d\mu < +\infty$, $\int |f|^s d\mu < +\infty$

only need to show for $r < p < s$.

Let $k = \frac{r}{p}\left(\frac{s-p}{s-r}\right) > 0$. $1-k = \frac{s}{p}\left(\frac{p-r}{s-r}\right) > 0$

Consider $\frac{p}{r}\cdot k + \frac{p}{s}(1-k) = 1$

$$\int |f|^p d\mu = \int |f|^{kp} |f|^{(1-k)p} d\mu$$

$$\leq \left(\int |f|^{kp\cdot\frac{r}{kp}} d\mu\right)^{\frac{kp}{r}} \left(\int |f|^{(1-k)p\,\frac{s}{p(1-k)}} d\mu\right)^{\frac{p(1-k)}{s}}$$

$$= \left(\int |f|^r d\mu\right)^{\frac{kp}{r}} \left(\int |f|^s d\mu\right)^{\frac{p(1-k)}{s}}$$

$< +\infty$

$\therefore |f| \in L^p$

Let $g: p \to \log \|f\|_p^p$

Want to show g is convex on $[r,s]$

consider for any $a, b \in [r,s]$, and $t \in (0,1)$

$$g(tb+(1-t)a) = \log \|f\|_{tb+(1-t)a}^{tb+(1-t)a}$$

$$= \log\left(\int |f|^{tb} |f|^{(1-t)a} \, d\mu\right)$$

$$\leq \log\left[\left(\int |f|^{tb \cdot \frac{1}{t}} \, d\mu\right)^t \cdot \left(\int |f|^{(1-t)a \cdot \frac{1}{1-t}} \, d\mu\right)^{1-t}\right]$$

$$= t \log\left(\int |f|^b \, d\mu\right) + (1-t) \log\left(\int |f|^a \, d\mu\right)$$

$$= \cancel{tb}\; t\, g(b) + (1-t)\, g(a)$$

$\therefore$ g is convex on $[r,s]$

By Cor. 4.12, g is continuous

Let $h(x) = \frac{1}{x}$, $m(x) = e^x$. both are continuous

Then. $m \circ (hg): p \to e^{\log \|f\|_p} = \|f\|_p$ is continuous

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