Math 190 Fall 2009
Midterm Examination Solutions

Problem 1 (6 pts) Compute

$$\int_1^e \frac{\sqrt{1+\ln x}}{x}dx.$$

Solution The integral can be solved using the substitution $u = 1 + \ln x$ since $du = dx/x$. This substitution gives us

$$\int \frac{\sqrt{1+\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}\sqrt{u^3} + C = \frac{2}{3}\sqrt{(1+\ln x)^3} + C.$$

So, the given integral is equal to

$$\frac{2}{3}\sqrt{(1+\ln x)^3}\Big|_1^e = \frac{2}{3}(\sqrt{8}-1).$$

Problem 2 (6 pts) Compute

$$\int \frac{\tan x + \sin x}{\cos^2 x}dx.$$

Solution We observe that

$$\int \frac{\tan x + \sin x}{\cos^2 x}dx = \int \frac{\sin x}{\cos^3 x}dx + \int \frac{\sin x}{\cos^2 x}dx.$$

Both of these integrals can be solved by the trigonometric substitution $u = \cos x$. So, the given integral is equal to

$$-\int \frac{du}{u^3} - \int \frac{du}{u^2} = \frac{1}{2u^2} + \frac{1}{u} = \frac{\sec^2 x}{2} + \sec x + C.$$

Problem 3 (6 pts) Determine if the improper integral exists. If so, evaluate it

$$\int_0^2 \frac{x}{x^2-1}dx.$$

Solution The integrand is discontinuous at $x = 1$ for $x \in [0, 2]$. So, the given integral will converge only if the improper integrals from 0 to 1 and 1 to 2 both converge. But both of these integrals diverge (see below) implying that the original integral diverges.

We show that the integral from 0 to 1 diverges to $-\infty$:

$$\int_0^1 \frac{x}{x^2-1}dx = \lim_{a\to 1}\int_0^a \frac{x}{x^2-1}dx = \lim_{a\to 1}\frac{\ln\left|x^2-1\right|}{2}\Bigg|_0^a = \lim_{a\to 1}\frac{\ln\left|a^2-1\right|-1}{2} = \lim_{b\to +0}\frac{\ln b - 1}{2} = -\infty.$$

Although this is enough to prove the divergence of the integral from 0 to 2 we also show that the integral from 1 to 2 diverges to $+\infty$:

$$\int_1^2 \frac{x}{x^2-1}dx = \lim_{a\to 1}\int_a^2 \frac{x}{x^2-1}dx = \lim_{a\to 1} \left.\frac{\ln\left|x^2-1\right|}{2}\right|_a^2 = \lim_{a\to 1}\frac{\ln 3 - \ln\left|a^2-1\right|}{2} = +\infty.$$

Problem 4 (6 pts) Compute

$$\int x \arctan x dx.$$

Solution The integral can be solved using integration by parts. If we set $u = x^2/2$ and $v = \arctan x$ then $du = xdx$ and $dv = dx/(x^2+1)$ and we get

$$\begin{aligned}\int x\arctan x dx &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx \\ &= \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C.\end{aligned}$$

Problem 5 (6 pts) Find an equation of the plane that contains the lines:

$$L_1: x = 1+t,\ y = t,\ z = t \quad \text{and} \quad L_2: x = 3+2s,\ y = 1+s,\ z = -1-s.$$

Solution An equation of a plane is given by $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$ where $\vec{n} = (a,b,c)$ is a normal vector to the plane and $\vec{r}_0 = (x_0, y_0, z_0)$ is a point in the plane. We can take $\vec{r}_0 = (1,0,0)$ since this point belongs to L_1 (set $t = 0$). The lines L_1 and L_2 are parallel to $(1,1,1)$ and $(2,1,-1)$, respectively, and so a normal vector $\vec{n}$ can be obtained as the vector product of these two vectors:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\vec{i} + 3\vec{j} - \vec{k}.$$

The equation of the plane is then equal to $-2x+3y-z = -2$.

Problem 6 (8pts) Let $P(1,1,2)$, $Q(1,0,1)$, $R(3,1,1)$ and let O be the origin.
(a) (4pts) Find the area of the triangle ΔPQR.

The area of the triangle is
$$A := 1/2|\overrightarrow{PQ} \times \overrightarrow{PR}|,$$
and
$$\overrightarrow{PQ} \times \overrightarrow{PR} = 1/2 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & -1 \\ 2 & 0 & -1 \end{vmatrix} = -\vec{i} - 2\vec{j} + 2\vec{k}.$$

The area equals the length of this vector, which $= 3$.

(b) (4pts) Find the volume of the parallelepiped with adjacent edges $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$.

The area of the parallelopiped is given by the scalar product, which is the determinant of the matrix:
$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$
which equals 3.

Problem 7 (6 pts) Find the line that intersects the line $L : x = t,\ y = 2t - 1,\ z = 2t + 1$ at a right angle and passes through the point $P(1,2,2)$.

We're looking for a line L which passes through the point $(1,2,2)$ and a point on the above line, which has the form $(t, 2t - 1, 2t + 1)$ for some t. These two points determine a line which must be orthogonal to the given line, whose direction vector is $v = (1,2,2)$.

A direction vector of the line L is $(1-t, 2-(2t-1), 2-(2t+1) = (1-t, 3-2t, 1-2t)$. To be orthogonal to the given line means that:
$$(1-t, 3-2t, 1-2t).(1,2,2) = 0 = 1 - t + 2(3-2t) + 2(1-2t) = 9 - 9t.$$

So, $t = 1$. The direction vector of L is therefore $w = (0,1,-1)$ and the equation of L is $(x,y,z) = (1,2,2) + t\overrightarrow{w} = (1, 2+t, 2-t)$.

Problem 8 (8 pts) A curve is described by the parametric equation:
$$x = 2\cos t + 1,\ \ y = 3\sin t.$$

(a) (4 pts) Use the parametric equation to find the slope of the tangent line to this curve, as a function of the parameter t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

and $dy/dt = 3\cos t$, and $dx/dt = -2\sin t$.

(b) (2 pts) Find the equation of the tangent line at $t = \pi/2$.

At $t = \pi/2$, the slope equals 0 so the line has equation $y = 3$.

(c) (2 pts) Show that the tangent line is vertical at $t = 0$ and explain this by sketching this curve.

At $t = 0$, the slope is infinite, and this can be seen from the sketch of the curve which shows that this is the equation of an ellipse:

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1.$$

Problem 9 (8 pts)
(a) (4 pts) For $0 \leq \theta < 2\pi$, sketch the (looping) curve $r = \sqrt{2} - 2\sin\theta$.

The curve has one inner loop and one outer loop. It looks similar to Figure 9.3.17 on page 679.

(b) (4pts) Find the area of the smallest loop enclosed by this curve.
The area equals

$$\frac{1}{2}\int_{\pi/4}^{3\pi/4} (\sqrt{2} - 2\sin\theta)^2 d\theta.$$

You will need to use the formula $\sin^2\theta = (1 - \cos 2\theta)/2$ to evaluate this integral. The answer is $\pi - 3$ but I did not penalize minor arithmetic errrors.

Problem 10 (8 pts) A rocket is fired from a height of 48 feet above sea level. The acceleration of the rocket, due to a combination of gravity and crosswinds, is given to be $\vec{a}(t) = 2\vec{i} + 4\vec{j} - 32\vec{k}$. The rocket is fired straight up at a speed of 32 ft/sec.
(a) (2pts) Express the initial velocity $\vec{v}(0)$ and initial position $\vec{r}(0)$ as vectors in 3-space.

$\vec{v}(0) = 32\vec{k}$ and $\vec{r}(0) = 48\vec{k}$.

(b) (4pts) At what time does the rocket return to the ground (at sea level)?

The rocket returns to the ground when the $\vec{k}$ component of the position vector is zero. We now need to find $\vec{r}(t)$. First, integrating the equation for $\vec{a}(t)$ we find that

$$\vec{v}(t) = 2t\vec{i} + 4t\vec{j} - 32t\vec{k} + \vec{v}(0) = 2t\vec{i} + 4t\vec{j} + (32 - 32t)\vec{k}.$$

Integrate $\vec{v}(t)$ to find $\vec{r}(t)$:

$$\vec{r}(t) = t^2\vec{i} + 2t^2\vec{j} + (32t - 16t^2)\vec{k} + \vec{r}(0) = t^2\vec{i} + 2t^2\vec{j} + (32t - 16t^2 + 48)\vec{k}.$$

Set the k component equal to zero to obtain $(t-3)(t+1) = 0$, which means that $t = 3$ seconds is the solution.

(c) (2pts) How far has the rocket traveled from its starting point? (You do not need to simplify your answer, i.e., work out the arithmetic.)

The starting point is $(0, 0, 48)$ and the ending point is $\vec{r}(3) = (9, 18, 0)$. The distance between them is $\sqrt{9^2 + (18)^2 + (48)^2}$.