MATH 1530 ABSTRACT ALGEBRA
PROBLEM SET 10, DUE TUESDAY APRIL 18 1PM IN CLASS

1. Let R be an integral domain, and let $f, g \in R$. Prove that $(f) = (g)$ if and only if $f = ga$ for some unit a.

   (Updated: extra credit) Also, give an example of a commutative ring R with identity $1 \neq 0$ such that the above does not hold.

2. Let K be a field, and consider the ring $K[[x]]$ of formal power series.
   (a) Prove that $K[[x]]$ is an integral domain.
   (b) Prove that the ideals of $K[[x]]$ are 0 or are of the form (x^n) for some integer $n \geq 0$.
   (c) Which of the ideals of $K[[x]]$ are principal? maximal? prime? Prove your answers.

3. Let $m, n \geq 1$ be integers. Express, with proof, the ideals in $\mathbb{Z}$

$$m\mathbb{Z} + n\mathbb{Z}, \quad (m\mathbb{Z})(n\mathbb{Z}), \quad m\mathbb{Z} \cap n\mathbb{Z}$$

   in the form $d\mathbb{Z}$ for some number d.

4. Dummit and Foote p. 257 problems 11, 12.

5. This is a problem in beginning algebraic geometry. Given an ideal $I \subseteq \mathbb{R}[x, y]$, we let the **v***anishing locus* or **v***ariety* of I be the subset of $\mathbb{R}^2$

$$V(I) = \{(a, b) \in \mathbb{R}^2 : f(a, b) = 0 \text{ for all } f \in I\}.$$

   (a) Prove that if $I = (f_1, \ldots, f_n)$ then $V(I) = \{(a, b) \in \mathbb{R}^2 : f_i(a, b) = 0 \text{ for all } i = 1, \ldots, n\}$.
   (b) Draw pictures of $V(I)$ for $I = (y(y - x^2))$ and for $I = (x - y, y - x^3)$.
   (c) Prove that if $I_1 \subseteq I_2$ are ideals of $\mathbb{R}[x, y]$ then $V(I_1) \supseteq V(I_2)$.
   (d) Using part (c) to help, prove the identities

$$V(I + J) = V(I) \cap V(J) \quad \text{and} \quad V(IJ) = V(I \cap J) = V(I) \cup V(J).$$

   Check your results in part (b) accordingly.

   For more, please see the book *Ideals, varieties, and algorithms* (Cox, Little, O'Shea).