MATH 1530 ABSTRACT ALGEBRA
PROBLEM SET 8, DUE TUESDAY APRIL 4 1PM IN CLASS

1. In class today (March 21), you formed groups of five students each with four students remaining, and then groups of seven students each with three students remaining. In your opinion, how many students came to class today?

2. Let $m, n \geq 1$ be integers. Prove that the set of numbers appearing as the order of some element of $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is
$$\{d \in \mathbb{Z}_{>0} : d \text{ divides } \operatorname{lcm}(m, n)\}.$$

3. Let G be a group and let
$$\Delta = \{(g, g) : g \in G\} \subseteq G \times G.$$
Prove that $\Delta \leq G \times G$. Furthermore, prove that Δ is normal if and only if G is abelian.

4. Let H, K be groups and let $\phi\colon K \to \operatorname{Aut}(H)$ be a homomorphism. Recall that $G = H \rtimes_\phi K$ contains subgroups
$$\widetilde{H} = \{(h, 1) : h \in H\} \trianglelefteq G \qquad \widetilde{K} = \{(1, k) : k \in K\} \leq G$$
isomorphic to H and K respectively. Prove that $\widetilde{K} \trianglelefteq G$ if and only if ϕ is trivial.

   (The map ϕ being *trivial* means that $\phi(k) = \mathrm{id}_H$ for all $k \in K$.)

5. Consider the group of rigid motions of $\mathbb{R}^2$, i.e. rotations, reflections, translations, and compositions of these. (Optional, ungraded: express this group as a nontrivial semidirect product.)

   For each of the following subsets S of $\mathbb{R}^2$, let G be the group of rigid motions of $\mathbb{R}^2$ that preserves S. Exhibit G as a nontrivial semidirect product in each case. Brief explanations may be helpful, but full proofs are not necessary.

   (a) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, the unit circle
   (b) $S = \mathbb{Z}^2$, the integer lattice points
   (c) $S = \{(x, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}$, an infinite horizontal strip.

6. Let p and q be distinct primes. Prove that any group of order pq is a semidirect product of two of its proper subgroups.