MATH 2050 ALGEBRAIC GEOMETRY FALL 2019
ALL PROBLEM SETS

Please submit hard copies, LaTeX strongly preferred, stapled, **before 5pm in my physical mailbox. Or by 9pm sharp in my email inbox.**

Problem Set 1. Due Monday September 16

Read Chapter 0 of Gathmann's notes. Hand in all the exercises from that chapter, other than 0.2.4.

These problems are "food for thought," and solutions that are less than rigorous are fine. I just want you to engage with the problems.

Problem Set 2. Due Monday September 23

Exercises from Chapter 1 of Gathmann's notes. Exercise 1.4.7 is optional.

Problem Set 3. Due Monday September 30

1. Fix $n \geq 1$. Let $X \subseteq \mathbb{A}^{2n}$ be the subset of $2 \times n$ matrices, as below, that have rank at most 1.
$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$
Consider the the *ideal of* 2×2 *minors*
$$J_{2,2} = \langle x_i y_j - x_j y_i : 1 \leq i < j \leq n \rangle \subset k[x_1, y_1, \ldots, x_n, y_n].$$
   (a) Verify that $X = Z(J_{2,2})$. (In fact $J_{2,2} = I(X)$ as well.)
   (b) Verify that X is irreducible. (One way is to exhibit X as the image under a continuous map of an irreducible space.)
   (c) On the open set U where not all y_i vanish, show $x_1/y_1 \in \mathcal{O}_X(U)$ and describe it explicitly as a function $U \to k$.
2. Consider the sheaf $\mathcal{C}$ on of continuous functions from (open sets of) the interval $[0, 1]$ to $\mathbb{R}$, and write $\mathcal{C}_0$ for the stalk of $\mathcal{C}$ at 0. Is $\mathcal{C}_0$ a local ring? Does it have nilpotents? Zero divisors?
3. *The skyscraper sheaf* at a point: let $x \in X$ any point in a topological space X, and A any ring. Verify that the assignment
$$\mathcal{F}(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{if } x \notin U \end{cases},$$
together with restriction maps id_A and 0 as appropriate, is a sheaf.
4. Vakil's notes Exercises 2.4.A, 2.4.B, 2.4.C.

Problem Set 4. Due Monday October 7

On your own: study Gathmann Lemmas 2.4.7, 2.4.10, and Exercise 2.6.13; these are various gluing statements.

1. Let $C = Z(x^2 + y^2 - 1)$ be the unit circle, and $U = C - \{(0,1)\}$. Let $f\colon U \to \mathbb{A}^1$ be "stereographic projection from the North pole to the y-axis." That is, let $f(p) = q$ for $(q,0)$ the unique point collinear with p and $(0,1)$.

   Show that f is a morphism of affine varieties. Does f extend to a morphism $C \to \mathbb{P}^1$?

2. Gathmann Exercises 2.6.6, 2.6.9, 2.6.10, 2.6.11.

3. Sheaves on a base: Prove Theorem 2.5.1 on p. 87 of Vakil's notes, in particular filling in Exercise 2.5.B.

Problem Set 5. Due Tuesday October 15 by 9pm sharp (email), or 5pm (hard copy)

1. For X any prevariety and Y an *affine* variety, prove that there is a canonical bijection

   $$\mathrm{Mor}(X, Y) \cong \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

   between morphisms $X \to Y$ and k-algebra homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Possible hint: Use Lemma 2.3.7 and glue.

2. Exercises 3.5.7, 3.5.8 from Gathmann's notes.

3. Work over $\mathbb{C}$. Fix integers d and r, and let

   $V_{d,r} = \{(a_0 : \cdots : a_d) \in \mathbb{P}^d \mid a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d = 0$ has at most r solutions $(x : y) \in \mathbb{P}^1\}$.

   (a) Prove that $V_{d,r}$ is an algebraic subset of $\mathbb{P}^d$. Give explicit equations $V_{d,r} = Z(I)$ cutting out $V_{d,r}$.

   (b) Your understanding of the previous part should be explicit enough to get `Macaulay2` to compute equations for $V_{5,3}$, say. Do this. You don't need to include the output since it should be long, as long as you give me correct input.

   (c) Determine, with proof, the number of irreducible components of $V_{d,r}$.

   (d) Can it happen that two irreducible components of $V_{d,r}$ have reducible intersection?

Note: An easy way is to access Macaulay2 is SageMathCell. `https://sagecell.sagemath.org/?lang=macaulay2` Here is sample code for you:

```
R=QQ[a..f];
M=matrix{{a,b,c},{d,2*e,3*f}};
minors(2,M)
```

Problem Set 6. Due Monday October 21 by 9pm sharp (email), or 5pm (hard copy)

1. Prove that *tangent space dimension is upper-semicontinuous* on any prevariety X. This means that for any $m \geq 0$, the set $\{p \in X \colon \dim T_p X \geq m\}$ is closed.

   (Reduce immediately to affine case; follow 4.4.8/4.4.9.)

2. Eisenbud-Harris exercises I-1, I-2. Just the answers are fine for these

3. Classify, with proof, the points of $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$, and describe the closure of each point in the Zariski topology.

4. Do the same for $\operatorname{Spec} (\mathbb{C}[x, y]/(xy))_{(x,y)}$.

5. Do the same for $\operatorname{Spec} \mathbb{Z}[x]$ (see Eisenbud-Harris II-37, II-38).

6. (added Friday) For the following ring maps $\phi \colon R \to S$, determine the corresponding maps $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ are, e.g., by describing what f does on each point of $\operatorname{Spec} S$.

   (a) The $\mathbb{C}$-algebra map $\mathbb{C}[t] \to \mathbb{C}[x, y]/(xy)$ sending t to $x + y$.

   (b) The natural map $k[x] \to \overline{k}[x]$, where $\overline{k}$ denotes algebraic closure.

Problem Set 7. Due Monday October 28 by 9pm sharp (email), or 5pm (hard copy)

1. Liu page 39-41 exercises 2.2.4, 2.2.8, 2.2.11, 2.2.12, and Vakil 2.7.B (or equivalently Liu 2.2.13)

Problem Set 8. Due Monday November 4 by 9pm sharp (email), or 5pm (hard copy)

1. We studied a once-punctured plane $\mathbb{A}^2_k - \{(0, 0)\}$ in class. Consider a *twice*-punctured plane. Is it isomorphic as a scheme to a once-punctured plane?

2. Eisenbud-Harris Exercise I-20.

3. Liu Exercises 2.3.14, 2.3.15.

4. updated 11/1: No new problems added, but start thinking about possible class presentation topics, e.g., the topics covered in Chapter 6 of Gathmann's notes would be great.

Problem Set 9. Due Monday November 11 by 9pm sharp (email), or 5pm (hard copy)

1. Vakil exercises 9.2.A, 9.2.B, 9.2.F

2. Eisenbud-Harris exercises II-11, II-12, II-14, II-15. See discussion on pp. 60-62 of that book.
3. (Optional) Sign up with me for a class presentation date and tentative topic. You should teach a topic of your choice from algebraic geometry, accessible to the class, for about 20 minutes.

Problem Set 10. Due Monday November 18 by 9pm sharp (email), or 5pm (hard copy)

1. Let $B = \operatorname{Spec} \mathbb{C}[t]$. Using a computer or otherwise, compute the limits of the following families of closed subschemes of $\mathbb{A}^n_B$ as $t \to 0$. Compute the primary decompositions of the ideals defining these limiting schemes.[1]
   (a) The plane curve $xy^2 = t$
   (b) Three concurrent lines becoming coplanar: the three lines through the origin and $(1, 0, 0), (0, 1, 0), (1, 1, t)$ respectively
   (c) Squashing a twisted cubic curve: the space curve whose closed points are (ts, s^2, s^3) for $t, s \in \mathbb{C}$.
2. Sanity check: Say X is an irreducible scheme with generic point η. If $\mathcal{O}_{X,\eta}$ is reduced does it follow that X is reduced?
3. Liu Exercises 2.4.3, 2.4.9, 2.4.11. For 2.4.3, if (and only if) you have never studied DVRs, you may take $\mathcal{O}_K = \mathcal{C}[\![t]\!]$.

Problem Sets 11 & 12. Due Friday December 6 by 9pm sharp (email), or 5pm (hard copy).

1. Practice the valuative criterion: use it to verify that $\mathbb{A}^n_k$ is proper over k if and only if $n = 0$.
2. Liu Exercises 2.5.3(b), 4.2.7. Extra: how do you add tangent vectors under the identification in 4.2.7?
3. Eisenbud-Harris III-15, III-16 on closed subschemes of Proj.
4. Eisenbud-Harris III-66 on examples of Hilbert polynomials.
5. **added:** Liu Exercises 5.1.5, 5.1.9(a), 5.1.12.

[1] Compute these in Macaulay2: `primaryDecomposition I`