The 4×4 minors of a $5 \times n$ matrix are a tropical basis

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joint work with Anders Jensen and Elena Rubei
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Background: Tropical Arithmetic

The tropical semiring $(\mathbb{R}, \oplus, \odot)$ consists of the real numbers equipped with tropical addition and multiplication:

$$\begin{aligned} x \oplus y &:= \min(x, y) \\ x \odot y &:= x + y. \end{aligned}$$

Example:

$$\begin{aligned} 3 \oplus 4 &= 3 \\ 3 \odot 4 &= 7 \end{aligned}$$

Background: Tropical Hypersurfaces

Let K be the field of well-ordered power series in a variable t

$$\{\alpha = \sum_{n \in S} a_n t^n \ : \ S \text{ a well-ordered subset of } \mathbb{R}, \ a \in \mathbb{C}\}.$$

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The tropicalization of a polynomial f with coefficients in K is the tropical polynomial F obtained by replacing each coefficient with its valuation (lowest exponent) and replacing all classical operations with tropical ones.

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The tropical hypersurface $T(f)$ of a polynomial $f \in K[x_1, \ldots, x_n]$ is the set of points in $\mathbb{R}^n$ at which F attains its minimum at least twice.

Example: $T(f)$ is a tropical line centered at $(-3, -1, 0)$.

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Theorem ("Fundamental Theorem of Tropical Geometry")

For $I \subseteq K[x_1, \ldots, x_n]$, the tropical variety $T(I)$ consists of those real points which lift (coordinate-wise) to the classical variety $V(I)$.

Definition 1: Tropical Rank

An $n \times n$ real matrix A is tropically singular if the minimum, over all permutations $\pi \in S_n$, of $a_{1\pi(1)} + \cdots + a_{n\pi(n)}$ occurs at least twice.

The tropical rank of a matrix is the size of its largest nonsingular square submatrix.

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Example: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ has tropical rank 2.

The set of $d \times n$ matrices of tropical rank $< r$ is the prevariety of the $r \times r$ minors of a $d \times n$ matrix.

Definition 2: Kapranov rank

Given a matrix $\mathcal{A}$ over the field K, let A be the real matrix of lowest exponents appearing in each entry of $\mathcal{A}$. We say that $\mathcal{A}$ is a lift of A.

Example: $\mathcal{A} = \begin{pmatrix} 1 & t & t^2 \\ 2t & 3t & 5t \\ 1+2t & 4t & 5t+t^2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

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The set of $d \times n$ matrices of Kapranov rank $< r$
is the variety of the $r \times r$ minors of a $d \times n$ matrix.

Outline of the Talk

These notions of rank were studied by Develin, Santos, Sturmfels; also Akian, Gaubert, Izhakian, Rowen, Kim-Roush, . . .

Today: Proof of a conjecture made by [Develin-Santos-Sturmfels]: the 4×4-minors of a $5 \times n$ matrix form a tropical basis

Tropical Rank versus Kapranov Rank

Question: Does every matrix of tropical rank $< r$ have Kapranov rank $< r$?

Equivalently: are the $r \times r$-minors of an $d \times n$ matrix a tropical basis? That is, are the prevariety and the variety of the $r \times r$ minors equal?

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- Yes, if $r \leq 3$ or $r = \min\{d, n\}$ (Develin, Santos, Sturmfels 2006)
- No, if $r = 4$ and $d = n = 7$ (Fano plane)
- Challenge posed for $r = 4, d = n = 5$ (50€, Berlin, 2007)

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Theorem

The 4×4-minors of a $5 \times n$ matrix are a tropical basis.

Computational proof for the 5×5 case

The tropical prevariety of the 25 4×4-minors is a pure 21-dimensional fan with 9-dimensional lineality space, and $f = (1450, 28450, 257300, \ldots, 2521800)$.

The tropical variety of the ideal $\langle 4 \times 4\text{-minors} \rangle$ is a pure 21-dimensonal fan with 9-dimensonal lineality space, and $f = (3250, 53650, 421750, \ldots, 2894400)$.

Same Euler characteristic $\chi = -3120$

Careful computations in `gfan` (Anders Jensen)
show that the supports agree.

Combinatorial Proof for a $5 \times n$ Matrix

Suppose

$$W = \begin{bmatrix} | & | & | & \cdots & | \\ w_1 & w_2 & w_3 & & w_n \end{bmatrix}$$

has tropical rank ≤ 3; want to lift it to a matrix in $K^{5\times n}$ of rank 3.

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Idea: Delete last row of W, get n coplanar points in $\mathbb{TP}^3$ They lie on a plane $a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus a_3 \odot x_3 \oplus a_4 \odot x_4$. So columns of W

lie on hyperplane

$$H_5 = a_1 \odot x_1 \oplus \cdots \oplus a_4 \odot x_4 \oplus \infty \odot x_5$$

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Similarly for other rows: Get five special hyperplanes $H_1, \ldots, H_5$.

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Lemma: If the *stable intersection* $H_i \cap_{\text{stab}} H_j$ of some pair contains W, then W lifts to a matrix of rank 3 as desired.

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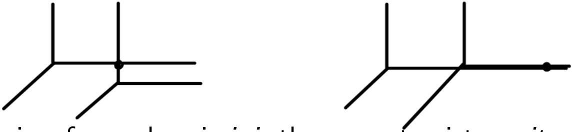

Otherwise, for each pair i, j, there must exist a *witness pair* k, l: a pair such that some column w_s lies in the closed sectors k and l, and no other closed sectors, for both hyperplanes H_i and H_j.

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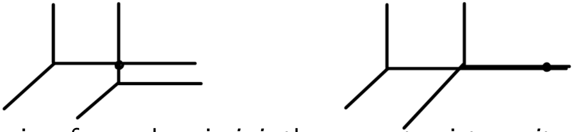

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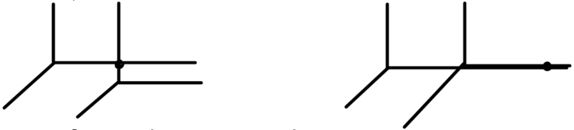

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In fact no tropical oriented matroid can satisfy these conditions (Ardila and Develin).

What next?

- 4×4-minors and 5×5-minors of $6 \times n$ matrices
- Topology, e.g. shellability, schönness of these spaces...
- Matrices with special structure: symmetric, Hankel, ...