A survey of the cycle double cover conjecture

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1 Introduction

A *cycle double cover* of a graph G is a list of cycles of G such that every edge of G appears exactly twice. The *cycle double cover conjecture* (CDC) is the following:

Conjecture 1. *Every bridgeless graph has a cycle double cover.*

This conjecture has been attributed variously to many mathematicians, but was known to be a consequence of the Strong Embedding Conjecture (Conjecture 2) by W. Tutte, G. Haggard, as well as by G. Szekeres and P. Seymour, amongst others [18], [16]. In what follows, we survey some of what is known about the above conjecture and discuss various related problems and techniques. We discuss the strong embedding conjecture and some variants in Section 2. We consider the structure of a minimal counterexample in Section 3. In Sections 4 and 5, we discuss generalizations to cycle k-covers and to integer combinations of cycles, respectively.

In preparing this document, the author found F. Jaeger's survey article [12] and to C.Q. Zhang's book [20] to be extremely helpful.

All of our graphs may contain loops and multiple edges.

2 Strong embeddings

We begin by considering some topological conjectures and their relations to cycle covers. By a *surface*, we mean a closed, connected, Hausdorff topological space, every point of which has an open neighborhood homeomorphic to $\mathbb{R}^2$. An *embedding* of a graph G on a surface S is a drawing of G on the surface without edge crossings. (This is an intuitive definition that can easily be made rigorous: see e.g. [19]). It is a *2-cell embedding* if every face is homeomorphic to an open disc. We call a 2-cell embedding *strong* if every face boundary is a cycle of the graph.

Conjecture 2. *[9] (Strong embedding conjecture) Every 2-connected graph has a strong embedding in some surface.*

Note that the strong embedding conjecture already implies CDC: given a bridgeless graph, embed each of its blocks into some surface. Then the face boundaries for each block, taken all together, constitute a cycle double cover for the original graph. In general, CDC is not known to imply the strong embedding conjecture, but for cubic graphs the two are indeed equivalent: given a list of cycles covering every edge twice, we "sew" a disc into each cycle. We need only check that each point of our space has a neighborhood homeomorphic to $\mathbb{R}^2$. This is clear for the interiors of the polygons and for the interiors of edges; for vertices, it follows from the fact that there is only one way to cover the three edges incident to a given vertex with three cycles.

Call a list of cycles in G an *orientable* cycle double cover of G if the cycles may be oriented so that each edge of G is used exactly once in each direction.

Conjecture 3. *Every 2-connected graph has a strong embedding in some orientable surface.*

Conjecture 4. *Every bridgeless graph has an orientable cycle double cover.*

Note that Conjecture 3 implies Conjecture 4: just traverse all cycles in the clockwise direction as given by the orientation of the surface. Furthermore, Conjecture 4 implies Conjecture 3 in the case of cubic graphs, as above.

We mention one more strengthening of CDC. An embedding of G is said to be *face-k-colorable* if its faces can be colored so that no two faces sharing an edge have the same color. An *m-cycle double cover* (generally, an *m-cycle k-cover*) is a list of m Eulerian subgraphs covering each edge exactly twice (generally, exactly k times). One can check that a graph has a 3-cycle double cover if and only if it has a 4-cycle double cover if and only if it has a nowhere zero 4-flow, and that the Petersen graph has none of these. However, we have the following conjectures.

Conjecture 5. *Every 2-connected graph has a face-5-colorable embedding in an orientable surface.*

Conjecture 6. *([3], [15]) Every bridgeless graph has a 5-cycle double cover.*

Conjecture 5 clearly implies Conjecture 6. A. Huck showed in [10] that a smallest counterexample to Conjecture 6 must have girth at least ten, and gave a polynomial time construction of a 5-cycle double cover for any bridgeless cubic graph with no Petersen minor.

Combining all of the above, we have:

Conjecture 7. *[12] Every 2-connected graph has a strong, face-5-colorable, embedding into an orientable surface.*

We remark that Conjecture 7, if true, would imply not only CDC but also Tutte's 5-flow conjecture:

Proposition 8. *Suppose G has a strong, face-5-colorable, orientable embedding. Then G has a nowhere-zero 5-flow.*

Proof. Given a coloring of the faces of the embedding with colors $\{0, 1, 2, 3, 4\}$, push a flow around the (oriented) boundary of each face, of magnitude equal to the color of that face. The result (mod 5) is a flow, and since the coloring was proper, it is nonzero on every edge. ☐

Thus, if Conjecture 7 is true, then one may apply Proposition 8 to the blocks of any bridgeless graph to obtain a nowhere zero 5-flow. Of course, Proposition 8 remains true with 5 replaced by any other number.

3 Smallest counterexamples

If CDC is false, then it must have a minimal counterexample. Following [12], we summarize some basic facts about the structure of such a graph in the proposition below. Recall that a graph is called *cyclically k-edge-connected* if every cut separating the graph into non-acyclic components has at least k edges. Call a cut *trivial* if one side consists of a single vertex. One may check that a bridgeless cubic graph is cyclically 4-edge-connected if and only if it has no nontrivial edge cuts of size 3.

Definition. A *snark* is a simple, cubic, cyclically 4-edge-connected graph with chromatic index 4.

Proposition 9. *Suppose G is a bridgeless graph not admitting a cycle double cover, with the minimum number of edges amongst all such graphs. Then G is a snark.*

Proof. If G had an edge cut of size 2, then then contracting one of those edges would yield a smaller counterexample to CDC. So G must be 3-edge-connected, and in particular each vertex has degree at least 3. Suppose some $v \in V(G)$ has $d(v) > 3$. Then by H. Fleischner's vertex-splitting lemma, there exist edges e, e' incident to v such that adding an edge between the ends of e and e' different from v and deleting e and e' yields a smaller bridgeless graph which must also be a counterexample to CDC, contradiction. Hence G is cubic. Now it is easy to check that G has no loops nor parallel edges.

Next, suppose G has a nontrivial 3-edge-cut. Then contracting each side into a single vertex in turn yields two smaller graphs, the cycle double covers of which can be pasted together to obtain a cycle double cover of G. So G has no nontrivial 3-edge-cuts, hence is cyclically 4-edge-connected.

Finally, suppose G is 3-edge-colorable, with color classes M_1, M_2, M_3. Then each M_i must be a perfect matching, so $M_1 \cup M_2$, $M_1 \cup M_3$, and $M_2 \cup M_3$ would be a 3-cycle double cover, contradiction. Hence G has chromatic index 4, by Vizing's Theorem.

□

L. Goddyn has shown that a smallest counterexample to CDC must have girth at least ten ([7], [8]). More recently, A. Huck improved this result to twelve, using a computer to check for certain reducible configurations [10]. One might hope, as was conjectured in [13], that snarks have bounded girth, but this is not the case: in [14], M. Kochol gave a construction of cyclically 5-edge-connected snarks of arbitrarily large girth.

4 Cycle k-covers

Recall that an m-cycle k-cover of a graph is a list of m Eulerian subgraphs using each edge exactly k times. We have concerned ourselves with the case $k = 2$; what can we say about other values of k? It is clear that for odd k, the graphs admitting a cycle-k-cover are the Eulerian graphs. For $k = 4$ and

$k = 6$, we have the following results, which are consequences of the 8-flow and the 6-flow theorems, respectively. For more on nowhere-zero flows, see, for instance, [20].

Theorem 10. *[2] Every bridgeless graph admits a 7-cycle 4-cover.*

Proof. Every bridgeless graph admits a nowhere zero $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$-flow, and hence has 3 Eulerian subgraphs, say C_1, C_2, C_3, whose union is $E(G)$. Then every edge of G appears precisely 4 times in the Eulerian subgraphs

$$\{C_1,\ C_2,\ C_3,\ C_1 \triangle C_2,\ C_1 \triangle C_3,\ C_2 \triangle C_3,\ C_1 \triangle C_2 \triangle C_3\}.$$

□

Remark. The proof of Theorem 10 just a special case of the following easily verified statement: given k sets $C_1, \ldots, C_k$, every element in their union occurs exactly 2^{k-1} times in the $2^k - 1$ sets

$$\{\triangle_{i \in I}\, C_i : \emptyset \neq I \subseteq \{1, \ldots, k\}\}.$$

For cubic graphs, Theorem 10 resembles the well-known Berge-Fulkerson Conjecture (see [6]), that every bridgeless cubic graph has 6 perfect matchings covering every edge exactly twice. Or, taking complements:

Conjecture 11. *Every bridgeless cubic graph has a 6-cycle 4-cover.*

Next, for $k = 6$, we have the following theorem, due to G. Fan.

Theorem 12. *[4] Every bridgeless graph admits a 10-cycle 6-cover.*

Proof. Let G be a bridgeless graph. Then G has a nowhere zero 6-flow, and hence a nowhere zero $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-flow. Thus, G has a nowhere zero 3-flow f and a nowhere zero 2-flow g such that $\{f \neq 0\} \cup \{g \neq 0\} = E(G)$. (The notation $\{f \neq 0\}$ stands for $\{e \in E(G) : f(e) \neq 0\}$), the "support" of f). Now $f, f+g$, and $f+2g$ are integer flows, so we may take their residues mod 3 and consider them as $\mathbb{Z}_3$-flows.

Now since $\{f \neq 0\}$ admits a nowhere zero $\mathbb{Z}_3$-flow, it admits a nowhere zero 3-flow, hence a nowhere-zero 4-flow, hence a nowhere zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow, hence has a 3-cycle double cover, say $\{C_{11}, C_{12}, C_{13}\}$. Similarly, let $\{C_{21}, C_{22}, C_{23}\}$ be a 3-cycle double cover of $\{f+g \neq 0\}$, and let $\{C_{31}, C_{32}, C_{33}\}$ be a 3-cycle double cover of $\{f + 2g \neq 0\}$. One can now check that

$$\big\{g \neq 0\big\} \cup \big\{\{g \neq 0\} \triangle C_{ij} : 1 \leq i, j \leq 3\big\}$$

is a 10-cycle 6-cover; just check the cases $g = 0$ and $g \neq 0$ separately. □

Combining the two theorems above, we see that, for each $k \geq 2$, every bridgeless graph has a $(3k + \lceil \frac{k}{3} \rceil)$-cycle $2k$-cover. Note, however, that the Berge Fulkerson Conjecture would imply the existence of a $3k$ cycle $2k$-cover for bridgeless cubic graphs and all even k.

5 Faithful cycle covers

Another generalization of CDC is the following: given an edge-weighting $w\colon E(G) \to \{0, 1, 2, \ldots\}$, when does there exist a list of cycles of G such that every edge e appears exactly $w(e)$ times? Call such a list a *faithful cycle cover* for w. Faithful cycle covers are well-studied; we summarize some results below.

Two necessary conditions for a faithful cycle cover are easily verified:

Proposition 13. *If the edge-weighting $w\colon E(G) \to \{0, 1, \ldots\}$ has a faithful cycle cover, then*

1. *The total weight across every edge cut is even, and*
2. *In every edge cut, no edge has weight more than half the total.*

These conditions are not sufficient in general: consider the Petersen graph with weight 2 on a perfect matching and 1 everywhere else. However, perhaps the following is true; if so, it implies CDC.

Conjecture 14. *[17] Let $w\colon E(G) \to \{1, 2\}$ satisfy the conditions in Proposition 13, and suppose that $\{e \in E(G) : w(e) = 1\}$ is a connected subgraph of G. Then w admits a faithful cycle cover.*

In any case, the conditions in Proposition 13 are sufficient for planar graphs, as shown by P. Seymour in [16]:

Theorem 15. *If G is a planar graph, then any function $w\colon E(G) \to \{0, 1, \ldots\}$ satisfying the conditions in Proposition 13 admits a faithful cycle cover.*

In fact, B. Alspach, L. Goddyn, and C.Q. Zhang proved the following strengthening.

Theorem 16. *[1] If G does not contain the Petersen graph as a topological minor, then any function $w\colon E(G) \to \{0, 1, \ldots\}$ that satisfies the conditions in Proposition 13 has a faithful cycle cover.*

Of course, if $w = 2$ on every edge, then the conditions in Proposition 13 hold for all bridgeless graphs, and we recover the cycle double cover conjecture. More generally, P. Seymour has conjectured:

Conjecture 17. *[16] If $w\colon E(G) \to \{0, 1, \ldots\}$ satisfies the conditions in Proposition 13, and in addition, w is even on every edge, then w admits a faithful cycle cover.*

Equivalently:

Conjecture 18. *[17] If $w\colon E(G) \to \{0, 1, \ldots\}$ is a nonnegative rational combination of cycles of G, then it is a nonnegative half-integer combination of cycles.*

Here is a possible approach to Conjecture 14.

Conjecture 19. *([17]) Let G be a bridgeless cubic graph and C a cycle of G. Then there is another cycle $C' \neq C$ of G with $V(C) \subseteq V(C')$.*

Note that if C is Hamiltonian, then Conjecture 19 follows from Smith's Theorem on Hamiltonian cycles.

We now show that Conjecture 19 implies Conjecture 14 and hence implies CDC.

Lemma 20. *If G is a minimal counterexample to Conjecture 14, then G is a bridgeless cubic graph with no nontrivial edge cuts of size 3.*

Proof. That G is bridgeless is clear. If G has a nontrivial edge cut of size 3, contract both sides of the cut in turn and obtain a cycle cover for G by pasting together cycle covers for the two smaller graphs.

Now suppose $v \in V(G)$ is a vertex of degree at least 4. We claim that we can split off any two edges e, e' incident to v, in the manner described in Proposition 9, to obtain a bridgeless graph G' with fewer edges. For if G' had a cut edge f, then $\{e, e'f\}$ was a nontrivial 3-edge-cut in G, contradiction. Now we may check that we can always find two edges incident to v such that splitting them off produces another graph in which $\{e \in E(G) : w(e) = 1\}$ is a connected subgraph of G. □

Proposition 21. *Conjecture 19 implies Conjecture 14.*

Proof. Let G be a graph and $w\colon E(G) \to \{1,2\}$ be a minimal counterexample to Conjecture 14. So G is bridgeless and cubic. Let $C = \{e \in E(G) : w(e) = 1\}$. Now C is an Eulerian connected subgraph of a cubic graph, so it is a cycle. By Conjecture 19, there exists another cycle $C' \neq C$ with $V(C) \subseteq V(C')$. Then $C \triangle C'$ is Eulerian, so let us subtract weight 1 from all edges in $C \triangle C'$ and delete edges of weight 0 to obtain a subgraph G' of G and a new edge weighting w' for G'. We claim that we can inductively apply Conjecture 14 to obtain a cycle w'-cover for G'. We need only check that that $\{w' = 1\}$ is Eulerian and connected, and that G' has no cut edges.

Indeed, the set $\{e \in E(G) : w'(e) = 1\}$ is precisely C', so is connected and Eulerian. Now suppose for a contradiction that $V(G')$ has a partition (X, Y) with a single cut edge e. Then $V(C')$ must lie entirely in one part, say X. But $V(C) \subseteq V(C')$ by choice of C', so $V(C)$ also lies inside X. But then no edges in $\delta(X, Y)$ were changed in getting G' from G, because none of those edges lie in C or C', so e was a cut edge in G as well as G', contradicting that G is bridgeless.

Thus, G' has a w'-cover, and adding weight 1 to the edges in $C \triangle C'$, we obtain a w-cover for G. □

We do not have a proof of Conjecture 19, but we can prove the following. Given a graph G, call a cycle C of G *central* if at most one component X of $G - C$ is such that $|\delta(X)|$ is odd.

Proposition 22. *Let G be a bridgeless cubic graph and let C be a central cycle of G. Then there exists a cycle $C' \neq C$ of G such that $V(C) \subseteq V(C')$.*

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