Tropical hyperelliptic curves

Melody Chan
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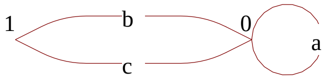
What is a tropical curve?

A **tropical curve** C is a triple (G, l, w), where G is a connected graph, $l : E(G) \to \mathbb{R}_{>0}$ is a length function, and

$$w : V(G) \to \mathbb{Z}_{\geq 0}$$

is a weight function on the vertices of G, with the property that every weight zero vertex has degree at least 3.



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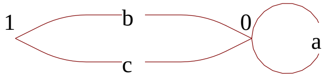

Its **genus** is $g(G) + \sum_{v \in V} w(v)$.

Motivation I: stratification of $\overline{\mathcal{M}_g}$ by dual graphs

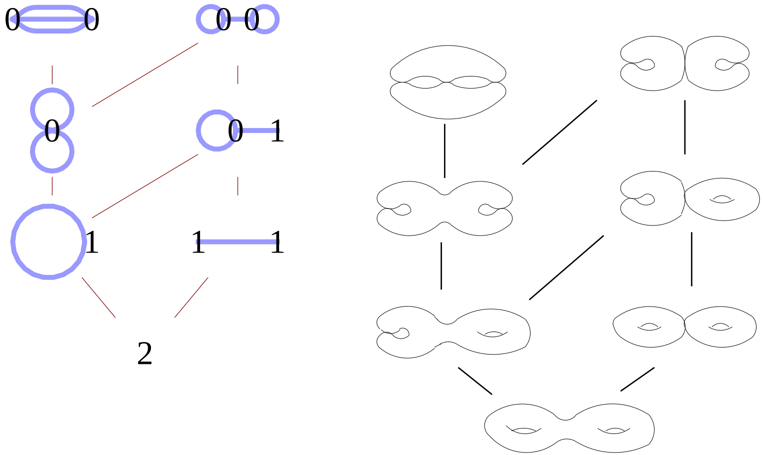

Figure: Posets of cells of M_2^{tr} (left) and of $\overline{\mathcal{M}_2}$ (right). Vertices record irreducible components, weights record genus, edges record nodes.

Motivation II: Berkovich spaces

Let K be an algebraically closed field, complete with respect to a nonarchimedean valuation $\mathrm{val} : K^* \to \mathbb{R}$ on it.

Examples: $\mathbb{C}_p$, completed Puiseux series.

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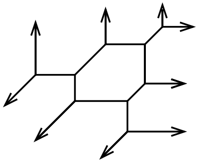
Suppose $X \subseteq (K^*)^n$ is an algebraic subvariety of the torus. Then the *tropicalization* of X is the set

$$\mathrm{Trop}(X) = \{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \in \mathbb{R}^n : (x_1, \ldots, x_n) \in X\}.$$



Note that $\mathrm{Trop}(X)$ is highly sensitive to the embedding of X.

Motivation II: Berkovich spaces

Let X be a smooth curve of genus $g \geq 1$ over K.

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X^{an} has a canonical deformation retract down to a finite metric graph Γ, decorated with some nonnegative integer weights, sitting inside it, called its **Berkovich skeleton**. In fact Γ is a tropical curve of genus g.

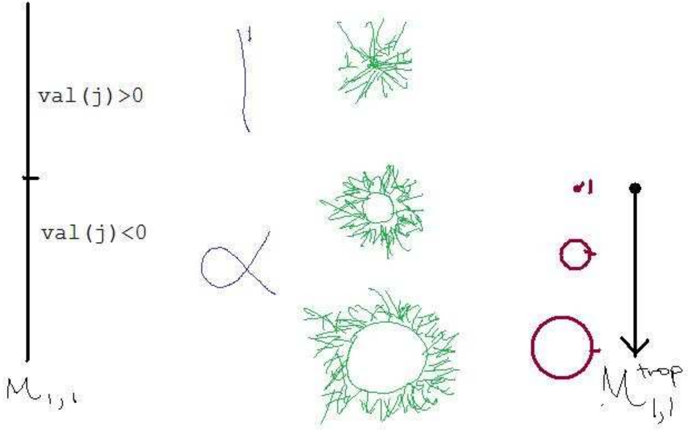
Furthermore, X^{an} is the inverse limit of all tropicalizations, and any finite piece of X^{an} can be found isometrically in some Trop(X) [Payne, Baker-Payne-Rabinoff].

Motivation II: Berkovich spaces

So we have a map

$$\mathcal{M}_g(K) \to M_g^{trop}$$

sending a curve X to its skeleton Γ. For example, elliptic curves:



Classical vs. tropical hyperelliptic curves

Let X be a complex algebraic curve of genus ≥ 2. Then TFAE:

1. There exists a divisor D on X with degree 2 and $dim|D| = 1$.
2. There exists an involution i such that X/i has genus 0.
3. There is a degree 2 holomorphic map $\phi : X \to \mathbb{P}^1$.

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More concretely, hyperelliptic curves are cut out by polynomials of the form

$$y^2 = h(x)$$

with h a polynomial of degree $2g + 1$ or $2g + 2$. Hence the space of hyperelliptic curves is $(2g - 1)$-dimensional.

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- In genus 3, every curve is either a plane quartic or it is hyperelliptic.
- Hyperelliptic loci are the smallest examples of *Brill-Noether loci*. [Caporaso, C-D-P-R, L-P-P]

Divisors on metric graphs [BN,GK,MZ]

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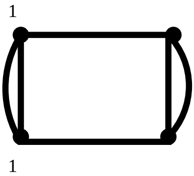

An effective divisor of degree 2.

Divisors on metric graphs [BN,GK,MZ]

A **rational function** on Γ is a continuous, piecewise affine-linear function $f : \Gamma \to \mathbb{R}$ with integer slopes.

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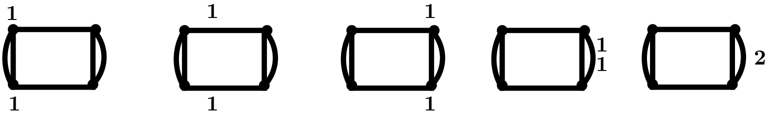
We say that two divisors D and D' are **linearly equivalent**, and we write $D \sim D'$, if $D' = \mathrm{div}(f) + D$ for some rational function f.

Divisors on metric graphs [BN,GK,MZ]

Equivalently, two divisors D and D' are **linearly equivalent** if D' can be obtained from D by a **chip-firing** procedure, as follows. Regard the coefficient of D at x as a number of chips at x, negative chips allowed. Pick any proper closed subset Z of Γ and send a chip down each edge leaving Z an equal distance; repeat.



Linearly equivalent divisors.

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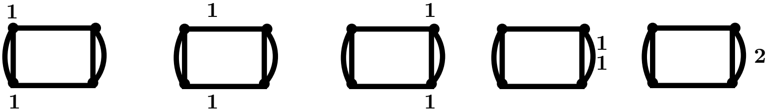
The **rank** $r(D)$ of a divisor D is defined to be

$$\max\{k \in \mathbb{Z} : \text{for all } E \geq 0 \text{ of degree } k,\ \exists E' \geq 0 \text{ with } D \sim E + E'\}.$$



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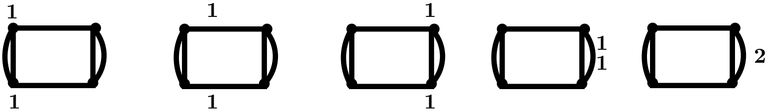

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The rank is the correct analogue for the dimension of the linear system $|D|$. With it, **Riemann-Roch** holds [BN]:

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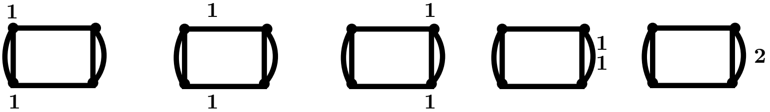

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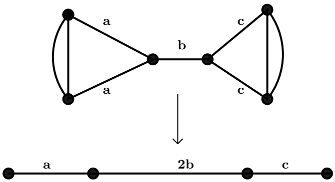
Definition

A metric graph is **hyperelliptic** if it has a divisor of degree 2 and rank 1.

Harmonic morphisms of metric graphs

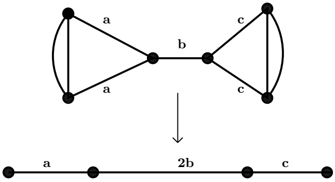
Next we define harmonic morphisms of metric graphs, which are analogues of holomorphic maps of Riemann surfaces.



A **morphism** of metric graphs $\phi : \Gamma \to \Gamma'$

1. sends vertices to vertices,
2. sends edges to edges (or collapses them down to vertices), in an incidence-preserving way;
3. if $\phi(e) = e'$ then $l(e')/l(e)$ is an integer. We call this number the **stretching factor** of e.

Harmonic morphisms of metric graphs



A morphism of metric graphs $\phi : \Gamma \to \Gamma'$ is **harmonic** if for all $x \in V(\Gamma)$, for all edges e' incident to $\phi(x)$, the sum of all stretching factors of edges above e' incident to x is independent of choice of e'.

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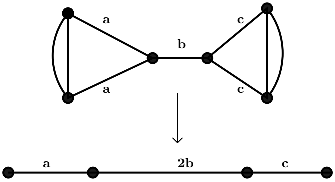

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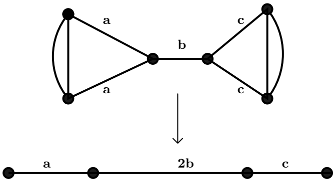
The **degree** of ϕ is the sum of all stretching factors above *any* edge of Γ'.

We say that ϕ is **nondegenerate** if every vertex $v \in V(G)$ is incident to some edge with a nonzero stretching factor.

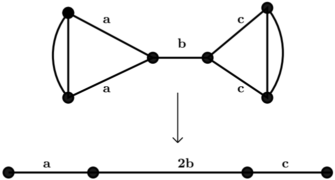


Harmonic morphisms are a good analogue of holomorphic maps of complex curves. For example, there is a natural way to define pushforwards and pullbacks [BN] such that the following holds:

Proposition

Let $\phi : \Gamma \to \Gamma'$ be a harmonic morphism of metric graphs. Then

1. $\phi^* \operatorname{div} f' = \operatorname{div} \phi^* f'$ *for any rational function* $f' : \Gamma' \to \mathbb{R}$.
2. $\phi_* \operatorname{div} f = \operatorname{div} \phi_* f$ *for any rational function* $f : \Gamma \to \mathbb{R}$.
3. $\deg \phi^* D' = \deg \phi \cdot \deg D'$ *for any divisor* D' *on* Γ'.
4. $\deg \phi_* D = \deg D$ *for any divisor* D *on* Γ.

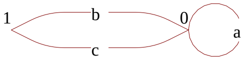


Theorem (C)

Let Γ be a metric graph of genus ≥ 2 with $|V(G)| > 2$ and no vertices of valence 1. Then TFAE:

1. *Γ is hyperelliptic, i.e. it admits a divisor with degree 2 and rank 1.*
2. *There exists an involution $i : \Gamma \to \Gamma$ such that Γ/i is a tree.*
3. *There exists a nondegenerate harmonic morphism of degree 2 from Γ to a tree.*

Tropical hyperelliptic curves of genus g



Recall: a tropical curve C is a triple (G, l, w), where G is a connected graph, $l : E(G) \to \mathbb{R}_{>0}$ is a length function, and $w : V(G) \to \mathbb{Z}_{\geq 0}$ is a weight function on the vertices of G, with the property that every weight zero vertex has degree at least 3.

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An tropical curve (G, w, l) is **hyperelliptic** if the metric graph obtained by adding $w(v)$ loops at each vertex $v \in V(G)$ is hyperelliptic.

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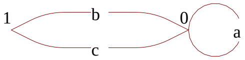

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The hyperelliptic algebraic curves of genus g form a $(2g-1)$-dimensional locus in $\mathcal{M}_g$. Next, we will prove a tropical analogue.

Tropical hyperelliptic curves of genus g

Theorem (C)

Let $g \geq 3$. The locus $H_g^{2,tr}$ of 2-edge-connected genus g tropical hyperelliptic curves is a $(2g-1)$-dimensional stacky polyhedral fan whose maximal cells are in bijection with trees on $g-1$ vertices with maximum valence 3.

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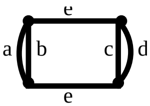
Proof.

Let us group genus g tropical hyperelliptic curves together according to their *combinatorial type*. That is, forget the edge lengths, but remember if two edges are required to have equal length.

Each combinatorial type is parametrized by a positive orthant modulo finite symmetries.



$$\frac{\mathbb{R}^5_{>0}}{S_2 \text{ wr } S_2}$$

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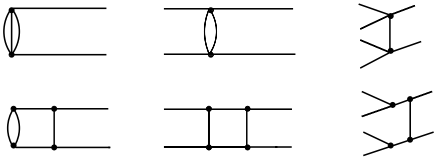

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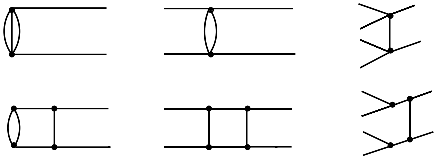

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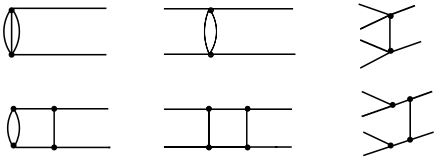

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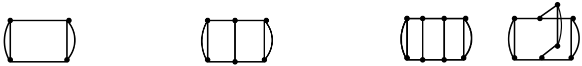

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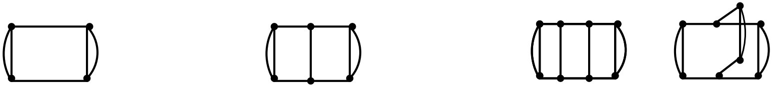
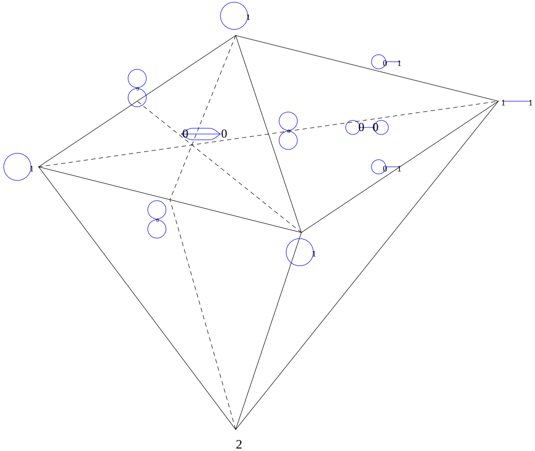
Corollary

Let $g \geq 3$. The number of maximal cells of $H_g^{(2),tr}$ is equal to the $(g-2)^{nd}$ term of the sequence

$$1, 1, 2, 2, 4, 6, 11, 18, 37, 66, 135, 265, \ldots$$



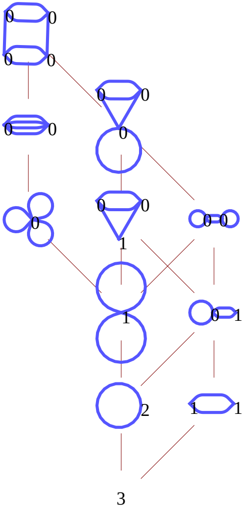
What do these spaces look like?



In the case $g = 2$ shown above, it is equal to the full moduli space M_g^{tr}. It consists of rational open polyhedral cones modulo symmetries, glued along boundaries via integral linear maps.

Theorem

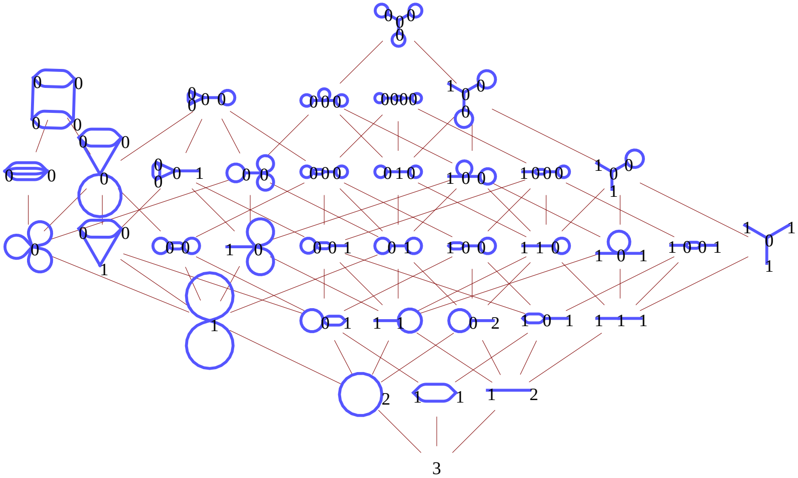
The moduli space $H_3^{2,tr}$ of 2-edge-connected tropical hyperelliptic curves has 11 cells and f-vector $(1, 2, 2, 3, 2, 1)$.



Theorem

The moduli space H_3^{tr} of tropical hyperelliptic curves has 36 cells and f-vector

$$(1, 3, 6, 11, 9, 5, 1).$$



Berkovich skeletons and tropical plane curves

Let X be a smooth hyperelliptic curve in the plane over a complete, nonarchimedean field K. Every such curve X is given by a polynomial of the form

$$P = y^2 + f(x)y + h(x)$$

for $f, h \in K[x]$.

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Proposition Trop(X) is dual to the **Newton subdivision** of Δ obtained by lifting each lattice point in Δ to height given by valuation of the coefficient in P, then taking lower faces of the convex hull.

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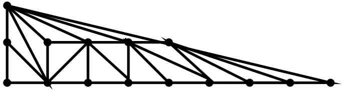
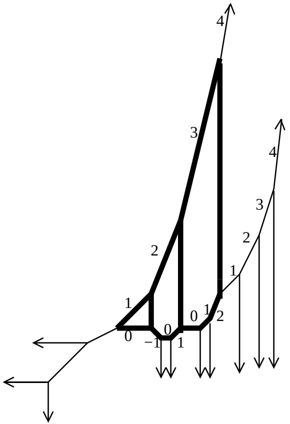
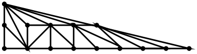
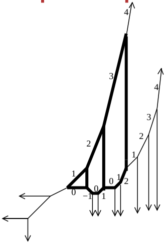




Berkovich skeletons and tropical plane curves





Theorem (C)

Let X/K be the curve defined by $P = y^2 + f(x)y + h(x)$, suppose the Newton complex of P is a unimodular triangulation, and suppose that the **core** *of* $\operatorname{Trop} X$ *is bridgeless.*

Then the minimal skeleton Σ of $\widehat{X}^{\mathrm{an}}$ is a standard ladder of genus g.

Here, $\operatorname{Trop}(X)$ is equipped with the *lattice length metric*, which normalizes the segment from $(0,0)$ to $(p,q) \in \mathbb{Z}^2$ to have length 1, if $\gcd(p,q) = 1$.

Further directions

- Study the map
$$\mathcal{M}_{g,n}(K) \to M_{g,n}^{trop}$$
and the behavior of Brill-Noether loci under this map. Every 2-edge-connected tropical hyperelliptic curve is the tropicalization of a hyperelliptic algebraic curve. The same is not true if we drop 2-edge-connectedness [AB,C].
- What about d-gonal curves, i.e. those admitting a divisor of degree d and rank 1?