Notes on Projective Geometry

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The purpose of these notes is to introduce projective geometry, to establish some basic facts about projective curves (the lowest dimensional kind of algebraic variety), and to sketch a proof of the famous Poncelet Porism. These notes have some exercises embedded in them. I wrote most of these notes for a class I taught a few years ago, which was essentially about elliptic curves. I put in some more stuff which people at ICERM might be interested in.

1 The Projective Plane

1.1 Basic Definition

For any field $\boldsymbol{F}$, the projective plane $P^2(\boldsymbol{F})$ is the set of equivalence classes of nonzero points in $\boldsymbol{F}^3$, where the equivalence relation is given by

$$(x, y, z) \sim (rx, ry, rz)$$

for any nonzero $r \in \boldsymbol{F}$. Let $\boldsymbol{F}^2$ be the ordinary plane (defined relative to the field $\boldsymbol{F}$.) There is an injective map from $\boldsymbol{F}^2$ into $P^2(\boldsymbol{F})$ given by

$$(x, y) \to [(x, y, 1)],$$

the equivalence class of the point $(x, y, 1)$. In this way, we think of $\boldsymbol{F}^2$ as a subset of $P^2(\boldsymbol{F})$.

A set $S \subset \boldsymbol{F}^3$ is called a *cone* if it has the following property: For all $v \in S$ and all nonzero $r \in \boldsymbol{F}$, we have $rv \in S$. Given a cone S, we define the *projectivization* $[S] \subset P^2(\boldsymbol{F})$ to be the set of points $[v]$ such that $v \in S$.

These notes will concentrate on the projective plane, but in general $P^n(\boldsymbol{F})$ has the same kind of definition. It is the space of equivalence classes of nonzero vectors in $\boldsymbol{F}^{n+1}$. Equivalently, $P^n(\boldsymbol{F})$ is the space of 1-dimensional subspaces of $\boldsymbol{F}^{n+1}$. To consider a special case, note that $P^3(\boldsymbol{R})$ is obtained from the sphere S^3 and identifying antipodal points. The idea here is that any nonzero vector in $\boldsymbol{R}^4$ is equivalent to a unit vector, and the only equivalent unit vectors are those at antipodal points. But, based on our understanding of the spin cover, we see that $SO(3)$ has the same description. Therefore $SO(3)$ and $P^3(\boldsymbol{R})$ are the same manifold!

1.2 Lines

A *line* in the projective plane is the set of equivalence classes of points in a 2-dimensional $\boldsymbol{F}$-subspace of $\boldsymbol{F}^3$. In other words, a line is the set of equivalence classes which solve the equation $ax + by + cz = 0$ for some $a, b, c \in \boldsymbol{F}$. That is, a line is the projectivization of a plane through the origin. The set of lines in $P^2(\boldsymbol{F})$ is often known as the *dual projective plane*. Think about it: Each line is specified by a triple (a, b, c), where at least one entry is nonzero, and the two triples (a, b, c) and (ra, rb, rc) give rise to the same lines.

Note that $P^2(\boldsymbol{F}) - \boldsymbol{F}^2$ is the line consisting of solutions to $z = 0$. This particular line is the *line at infinity* and we sometimes write it as L_∞.

Exercise 1: Prove that every two distinct lines in $P^2(\boldsymbol{F})$ intersect in a unique point. Likewise, prove that every two distinct points in $P^2(\boldsymbol{F})$ are contained in a unique line.

Exercise 2: Let $\boldsymbol{F}$ be a finite field of order $N = p^n$. How many points and lines does $P^2(\boldsymbol{F})$ have.

The cross product is useful for computing in the projective plane. We can represent a point by a vector $V \in \boldsymbol{F}^3$. The line through two points represented by V and W is represented by $V \times W$. Likewise, we can represent lines by vectors. The intersection of the lines represented by V and W is represented by $V \times W$. From this description, you see that the two operations of *joining* two points by a line and *meeting* two lines in the intersection point are completely interchangeable. Note that $V \times W$ is always nonzero if V and W are not multiples of each other. You can use this approach to help with Exercise 1 above.

1.3 Projective Transformations

A linear isomorphism from $\boldsymbol{F}^3$ to itself respects equivalence classes, and therefore induces a map from $P^2(\boldsymbol{F})$ to itself. This map is called a *projective transformation*. A projective transformation is always a bijection which maps lines to lines. In case $\boldsymbol{F} = \boldsymbol{R}$ or $\boldsymbol{F} = \boldsymbol{C}$, the projective transformations are continuous. The set of projective transformations forms a group, often known as *the projective group*. The Lie group of projective transformations is denoted $PGL_3(\boldsymbol{F})$. At least when $\boldsymbol{F} = \boldsymbol{R}$, this group is 8 dimensional. There is a unique projective transformation of $\boldsymbol{P}^2(\boldsymbol{R})$ which maps any quadruple of general position points to any other quadruple of general position points.

The projective transformations have the beautiful property that they maps lines to lines. This derives from the fact that an invertible linear transformation maps a 2-dimensional vector subspace to a 2-dimensional vector subspace. At least over the reals and complex numbers, projective transformations are characterized by this property.

Exercise 3: Let $h : P^2(\boldsymbol{R}) \to P^2(\boldsymbol{R})$ be a homeomorphism which maps lines to lines. Prove that h is a projective transformation.

Projective transformations mimic our vision. Suppose that we draw a picture on a square, and then we hold up the square obliquely and look at it. The resulting picture is the image of the one on the square under a projective transformation. You can use this principle to draw in perspective. For instance, suppose that a little red spot is at the center of the square. This spot is the place where the two diagonals of the square intersect. To locate the point on the quadrilateral you see, intersect the diagonals of the quadrilateral and that is where you should put the red spot.

Projective transformations do not preserve distances, but they do preserve something called the *cross ratio*. Given 4 points $a, b, c, d \in \boldsymbol{F}$, we define

$$[a, b, c, d] = \frac{(a-c)(b-d)}{(a-b)(c-d)} \tag{1}$$

Here we are identifying $\boldsymbol{F}$ with the subset of the affine patch of the form $(x, 0, 1)$. The cross ratio is invariant under any projective transformation which preserves $\boldsymbol{F}$. This lets us define the notion more generally. Given any 4 collinear points in $P^2(\boldsymbol{F})$, all on a line L, we can find projective transformation which maps (A, B, C, D) to points (a, b, c, d) as above. We then define $[A,B,C,D]=[a,b,c,d]$. From what we have said about invariance, this value is indepenent of any choices we make in mapping L to $\boldsymbol{F}$.

The cross ratio can also be defined as follows. Represent the points A,B,C,D by vectors of the same name. Then consider the quantity

$$\frac{(A\times C)*(B\times D)}{(A\times B)*(C\times D)}. \tag{2}$$

Here $(*)$ denotes pointwise multiplication:

$$(a_1,a_2,b_2)*(b_1,b_2,b_3)=(a_1b_1,a_2b_2.a_3.b_3).$$

The quantity in Equation 2 has the form (x,x,x), and the quantity x is the cross ratio.

1.4 The Hilbert Metric

Now we specialize to the real projective plane $P^2(\boldsymbol{R})$. A set $C\subset P^2(\boldsymbol{R})$ is *convex* if there is a projective transformation T such that $T(C)\subset \boldsymbol{R}^2$, the affine patch, and $T(C)$ is convex in the ordinary sense. When C is open and convex, there is a canonical metric on C called the *Hilbert metric*. Given $x,y\in C$ we define

$$d(x,y)=\log([w,x,y,z]) \tag{3}$$

Here w,y are the two points on ∂C such that w,x,y,z are collinear, on a line L, and appear in order in the line segment $L\cap C$.

The Hilbert metric is projectively natural. If $T:C\to C$ is the restriction of a projective transformation, then T is an isometry. As a special case, the Hilbert metric on the open unit disk gives a metric which is isometric to the Klein model of the hyperbolic plane.

Exercise 4: Prove that the Hilbert metric is indeed a metric on any open convex set C. Hint: The triangle inequality is the interesting step. Use monotonicity properties of the cross ratio to reduce to the case of hexagons.

Though we are concentrating on the projective plane, we note that the cross ratio also makes sense in projective space, and one can define a metric on any open convex subset of $P^n(\boldsymbol{R})$. When you do this for the open unit ball, you get hyperbolic space of the appropriate dimension.

Exercise 5: Contemplate the Hilbert metric on an open equilateral triangle. This metric in this case is also known as the *Hex metric*. More generally contemplate the Hilbert metric on convex polygons. What is it like? What properties does it have? (Hint: ask Daryl Cooper.)

2 Homogeneous Polynomials

2.1 Basic Definition

Given a triple $I = (a_1, a_2, a_3)$, we define

$$X^I = x_1^{a_1} x_1^{a_2} x_3^{a_3}. \tag{4}$$

Here a_1, a_2, a_3 are non-negative integers. We define $|I| = a_1 + a_2 + a_3$. We say that a *homogeneous polynomial* of degree d (in 3 variables) over the field $\boldsymbol{F}$ is a polynomial of the form

$$\sum_{|I|=d} c_I X^I, \qquad c_I \in \boldsymbol{F}. \tag{5}$$

The variables here are x_1, x_2, x_3. Sometimes it is convenient to use the variables x, y, z in place of x_1, x_2, x_3.

Exercise 6: Let P be a degree d homogeneous polynomial and let T be a projective transformation. Prove that $P \circ T$ is another homogeneous polynomial of degree d.

2.2 Homogenization and Dehomogenization

A degree d polynomial in 2 variables has a *homogenization*, where we just pad the polynomial with suitable powers of the third variable to get something that is homogeneous. An example should suffice to explain this.

$$x^5 + 3x^2y^2 + x^2y - 5 \qquad \Longrightarrow \qquad x^5 + 3x^2y^2z + x^2yz^2 - 5z^5.$$

Conversely, every homogeneous polynomial of degree d in 3 variables has a *dehomogenization*, obtained by setting the third variable to 1. The operations of homogenization and dehomogenization are obviously inverses of each other.

2.3 Projective and Affine Curves

Let P be a homogeneous polynomial of degree d. If $v \in \boldsymbol{F}^3$ and $r \in \boldsymbol{F}$, we have

$$P(rv) = r^d P(v). \tag{6}$$

Therefore, when $r \neq 0$, we have $P(rv) = 0$ if and only if $P(v) = 0$. In other words, the solution $P = 0$ is a cone in $\boldsymbol{F}^3$. Because of this fact, the following definition makes sense.

$$V_P = \{[v]|\ P(v) = 0\} \subset P^2(\boldsymbol{F})\}. \tag{7}$$

This V_P is just the projectivization of the solution set $P = 0$. The set V_P is known as a *projective curve*.

A projective curve is a kind of completion of the solution set to a polynomial. Suppose that $p(x, y)$ is a degree d polynomial in 2 variables and $P(x, y, z)$ is the homogenization. Let $V_p = \{(x, y)|\ p(x, y) = 0\}$. The set V_p is known as an *affine curve*. Since $\boldsymbol{F}^2$ is naturally a subset of $P^2(\boldsymbol{F})$, in the way described above, we have the inclusion

$$V_p \subset \boldsymbol{F}^2 \subset P^2(\boldsymbol{F}). \tag{8}$$

Exercise 7: Interpreting V_p as a subset of V_P, prove that $V_p = V_P \cap \boldsymbol{F}^2$. So, the projective curve V_P is obtained from V_p by adjoining the points of $P^2(\boldsymbol{F}) - \boldsymbol{F}^2$ where P vanishes.

2.4 Nonsingular Curves

It makes sense to take the formal partial derivatives of a polynomial over any field. In particular, the *gradient*

$$\nabla P = \left(\frac{dP}{dx}, \frac{dP}{dy}, \frac{dP}{dz}\right) \tag{9}$$

makes sense. We say that a *singular point* of P is a point $v \neq 0$ such that $P(v) = 0$ and $\nabla P(v) = 0$. If $r \in \boldsymbol{F}$ is nonzero, then v is a singular point if and only if rv is a singular point. The polynomial P is called *nonsingular* if it has no singular points. The projective curve V_P is called nonsingular if P is nonsingular.

When it comes time to discuss elliptic curves, we will always work with nonsingular ones.

Exercise 8: Suppose that V is a nonsingular projective curve and T is a projective transformation. Prove that $T(V)$ is also a nonsingular projective curve. This is kind of a painful exercise in the chain rule.

2.5 The Tangent Line

Let P be a nonsingular projective curve and let $[v] \in P^2(\boldsymbol{F})$ be a point. The *tangent line* to P at $[v]$ is defined to be the line determined by the equation

$$\nabla P(v) \cdot (x, y, z) = 0. \tag{10}$$

This is a line through the origin. In case $\boldsymbol{F} = \boldsymbol{R}$ you can think about this geometrically. In $\boldsymbol{R}^3$, the tangent plane to the level set $P(x, y, z) = 0$ at the point (x_0, y_0, z_0) is given by the equation

$$((x, y, z) - (z_0, y_0, z_0)) \cdot \nabla P = 0.$$

Here we are assuming that $P(x_0, y_0, z_0) = 0$.

Since P is a homogeneous polynomial, $P = 0$ along the line through (x_0, y_0, z_0). This means that $\nabla P(x_0, y_0, z_0) \cdot (x_0, y_0, z_0) = 0$. (This works in any field, but it requires an algebraic proof in general.) Therefore, in this case, the equation of the tangent plane simplifies to

$$(x, y, z) \cdot \nabla P = 0.$$

So, in $\boldsymbol{R}^3$ the plane Π_0 given by Equation 10 is a good approximation along the line through (x_0, y_0, z_0) to the level set $P(x, y, z) = 0$. Both sets are cones, and so the projectivization of the tangent plane (the tangent line) is a good approximation of the projectivization of the polynomial level set (the projective curve).

Exercise 9: Let $f(x, y)$ be a polynomial in 2 variables, and let $P(x, y, z)$ be its homogenization. Let (x_0, y_0) be some point where $f(x_0, y_0) = 0$ and $\nabla f(x_0, y_0) \neq 0$. We think of (x_0, y_0) as a point of $P^2(\boldsymbol{R})$ by identifying it with $[x_0, y_0, 1]$, as above. Prove that the tangent line to the level set of f at (x_0, y_0) is exactly the projectivization of the plane given by Equation 10. In other words, reconcile the definition of tangent line given above with the usual definition given in a calculus class. This is also a bit tedious.

3 The Poncelet Porism

Let E_0 and E_1 be two ellipses, with E_0 contained in the interior of the region bounded by E_1 as shown in Figure 1 below. We call a polygon P *sandwiched* between E_0 and E_1 if the vertices of P are contained in E_1 and the edges of P are tangent to E_0. One would also say that P is inscribed in E_1 and circumscribed about E_0. Poncelet's porism says that if some P exists, there is in fact a continuous family P_t of such polygons, such that $P = P_0$. The family P_t sort of circulates around E_0 and E_1, except that the shapes subtly change as the family moves around. In general, the polygons in the family are not projectively equivalent. This fact makes the Poncelet Porism deeper than the Steiner Porism. (The Steiner Porism makes a similar statement about necklaces of circles all tangent to two circles and is much easier to prove.)

Here I'll present a fairly classical proof of the Poncelet porism. This proof is similar to the one found in the following reference:

P. Griffiths and J. Harris, *A Poncelet Theorem in Space*, Comment Math Helvetici, **52** (1977) pp 145-160

The proof I present is exerpted pretty much word for work from a paper I wrote entitled *The Poncelet Grid*. This proof is a bit terse, so ask me about it if you have questions.

We can apply a projective transformation and normalize so that E_0 is the unit circle and

$$E_1 = \{(x, y)|\ (x/a)^2 + (y/b)^2 = 1\}, \tag{11}$$

where $a > b > 1$. We ignore the relatively trivial case when $a = b$. Actually, this normalization is not so important for the sketch we give here, but in the paper I used it for other purposes. The main advantage is that the normalization makes some of the equations more concrete.

We give E_0 the counterclockwise orientation. At each point $p \in E_0$ the tangent ray to E_0 at p intersects E_1 in a point q. There is a unique point $r \in E_0$ such that $r \neq p$ and the ray $\overrightarrow{qr}$ contains the tangent ray to E_0 at r. Figure 1 shows the situation.

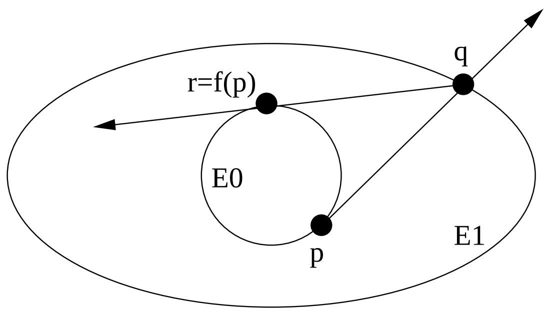

Figure 1

We define $f : E_0 \to E_0$ by $f(p) = r$. The point p is contained in an edge of a Poncelet n-gon if and only if $f^n(p) = p$. Poncelet's porism says, given two points $p, p' \in E_0$, we have $f^n(p) = p$ if and only if $f^n(p') = p'$. This result is proved by showing that f is conjugate to a rotation of a circle.

Here we sketch (a variant of) the classical argument. We change the notation somewhat and let $P(\boldsymbol{C})$ denote the complex projective plane. We make this change so that we can gracefully denote the dual plane as $P^*(\boldsymbol{C})$. Let $E_0(\boldsymbol{C})$ and $E_1(\boldsymbol{C})$ be the conics in $P(\boldsymbol{C})$ which extend E_0 and E_1. In other words, we just take the equations for E_0 and E_1 and solve them over the complex numbers. There are 4 complex lines which are simultaneously tangent to $E_0(\boldsymbol{C})$ and $E_1(\boldsymbol{C})$.

Lemma 3.1 *The* 4 *complex lines simultaneously tangent to* E_0 *and* E_1 *have the form* $\pm icx \pm dy = 1$*, where*

$$c^2 = \frac{b^2 - 1}{a^2 - b^2}; \qquad d^2 = c^2 + 1. \tag{12}$$

Proof: Let $P^*(\boldsymbol{C})$ denote the dual projective plane. There is a projective duality $\delta : P(\boldsymbol{C}) \to P^*(\boldsymbol{C})$ which carries the line $ax + by = 1$, considered as a subset of $P(\boldsymbol{C})$, to the point $(a, b) \in P^*(\boldsymbol{C})$. Let $S(E_j)$ denote the set of lines tangent to $E_j(\boldsymbol{C})$. Then $\delta(S(E_0))$ is the conic having equation $x^2 + y^2 = 1$ and $\delta(S(E_1))$ is the conic having equation $(ax)^2 + (by)^2 = 1$. The intersection points of these dual conics are $(\pm ic, \pm d)$, where c and d are as in Equation (12). The result follows immediately. ♠

Let T denote the set of pairs (q, l) where $q \in E_1(\boldsymbol{C})$ and l is a complex line containing q and tangent to $E_0(\boldsymbol{C})$. We define $\pi : T \to E_0(\boldsymbol{C})$, by the equation $\pi(q, l) = l \cap E_0(\boldsymbol{C})$. The complex line l_p, tangent to $E_0(\boldsymbol{C})$ at p, intersects $E_1(\boldsymbol{C})$ in either 1 or 2 points, depending on whether or not l_p is also tangent to $E_1(\boldsymbol{C})$. Hence π is a double branched cover, branched at 4 points. This forces T to be a torus. The inclusion $T \hookrightarrow P(\boldsymbol{C})$ gives a complex structure on T in which π is a holomorphic homeomorphism.

Like all complex tori, T has a Euclidean metric, unique up to scale, in which all holomorphic and anti-holomorphic self-homeomorphisms are isometries. This is the uniformization theorem for elliptic curves. There are two natural involutions on T:

- We have $i_1(q, l) = (q, l')$, where (generically) l' is the other line through q which is tangent to $E_0(\boldsymbol{C})$. The map i_1 has 4 fixed points; these are pairs (q, l) where $q \in E_0(\boldsymbol{C}) \cap E_1(\boldsymbol{C})$.

- We have $i_2(q, l) = (q', l)$ where (generically) q' is the other point of $l \cap E_1(\boldsymbol{C})$. The map i_2 has 4 fixed points; these are the pairs (q, l) where l is tangent to both $E_0(\boldsymbol{C})$ and $E_1(\boldsymbol{C})$.

The fixed points of i_1 are completely distinct from the fixed points of i_2. Hence $\widetilde{f} := i_1 \circ i_2$ acts as a translation−i.e. with no fixed points.

The set $\pi^{-1}(E_0)$ consists of two circles, $\widetilde{E}_0$ and $\widetilde{E}_0'$. We label so that $\widetilde{E}_0$ consists of elements (q, l) where l contains a ray tangent to E_0, pointing in the counterclockwise direction, and q is on this ray. (Compare Figure 1.) The map π intertwines the action of $\widetilde{f}$ on $\widetilde{E}_0$ with the action of f on E_0. Complex conjugation preserves both $\widetilde{E}_0$ and $\widetilde{E}_0'$ and hence induces an anti-holomorphic isometry of T. The fixed point set is exactly $\pi^{-1}(E_0)$. Since $\widetilde{E}_0$ is one component of the fixed point set of an isometry, $\widetilde{E}_0$ is a closed geodesic on T. All in all, $\widetilde{f}$ is a free isometry of $\widetilde{E}_0$, which is to say, a rotation. This proves that f is conjugate to a rotation.