The Saul Sol Challenge: Sol is the Riemannian manifold whose underlying space is $\boldsymbol{R}^3$ and whose Riemannian metric is given by. $e^{-2z}dx^2+e^{2z}dy^2+dz^2$. This Riemannian metric induces a path metric on Sol in the usual way. Saul Schleimer's challenge is as follows: Compute the Sol metric distance from $(0,0,0)$ to $(2\pi,2\pi,0)$ up to 5 digits of accuracy.

A Solution: Matei Coiculescu and I (Rich Schwartz) figured this out together. The solution depends on the reliability of Mathematica.

It follows from the work in our paper, or indeed in Matt Grayson's thesis, that the vector V_a corresponding (*via* the Riemannian exponential map) to a length minimizing Sol geodesic segment connecting $(0,0,0)$ to $(H_a,H_a,0)$ has the form $V_a = L_a(a,a,\sqrt{1-2a^2})$. Here $a \in (0,\sqrt{2}/2]$. Based on the formulas in Grayson's thesis and in Troyanov's paper, we worked out (and also numerically tested) the formulas

$$L_a = \sqrt{8+8m}K(m), \quad H_a = \frac{4E(m)}{\sqrt{1-m}} - \sqrt{4-4m}K(m), \qquad m = \frac{1-2a^2}{1+2a^2}.$$

Here K and E are respectively the complete elliptic integrals of the first and second kind, called `EllipticK` and `EllipticE` in Mathematica. The parameter $m \in [0,1)$ is the same as what is used in Mathematica.

We write $L(m) = L_a$ and $H(m) = H_a$, with m and a as above. We have $L(0) = \pi\sqrt{2}$ and $H(0) = \pi$. The functions $L(\cdot)$ and $H(\cdot)$ are monotone increasing in m and tend to ∞ as $m \to 1$. So, there is a unique $m_0 \in (0,1)$ such that $H(m_0) = 2\pi$.

We use a bisection algorithm to trap m_0 in a very small interval. We define $\beta[a,b]$ to be either $[a,c]$ or $[c,b]$ according as $H(c) > 2\pi$ or $H(c) \leq 2\pi$. Here we have set $c = (a+b)/2$. We check that $H(1/2) < 2\pi < H(3/4)$. We compute that $\beta^{98}[1/2,3/4] = [m_-,m_+]$, where

$$m_- = 2^{-100}(832071997981534918194250276977); \qquad m_+ = m_- + 2^{-100}.$$

By construction $m_0 \in [m_-,m_+]$. Hence $L(m_0) \in [L(m_-),L(m_+)]$.

We note that $L(m_+) - L(m_-) = 5.404... \times 10^{-30}$ and so the first 25 digits of $L(m_0)$ are the same as the first 25 digits of $L(m_\pm)$. We compute

$$L(m_\pm) = 7.33740\ 11575\ 37037\ 17875\ 58978...$$

This gives the first 25 digits of the length of the minimal geodesic in Sol connecting $(0,0,0)$ to $(2\pi,2\pi,0)$.