

Set Theory Axioms

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1 Why Does Mathematics Work?

Mathematics seems to work amazingly well compared to other areas of human thought. If you read the papers, you know that economists, physicians, politicians, historians, etc., make one mistake after another. For a while it seems that a glass of wine every day is good for you, and then somewhat later the recommendation is reversed. For a while it seems that “trickle down economics” might be a good idea, then it seems bad, then some people embrace it again. And so on. This rarely happens with mathematics. A theorem that has been completely understood is never overturned.

The hard sciences somewhat rival mathematics in accuracy, but this seems to be a function of the extent to which mathematical models actually predict what happens in the world. Newton’s theory of mechanics is flawless from a mathematical point of view, but it describes an idealized universe that turns out to be a bit different from ours. Physical laws have an *asymptotically correct* feel to them. They make better and better predictions as we refine our knowledge of the external world. Mathematics seems to be correct all along.

So, why does mathematics work so well? This seems to be a hard question to answer, and it has been a topic of philosophy for centuries. The best answer I can give is that mathematics deals with very simple objects, and reasons about them according to a small collection of precise logical methods which appear to always lead to the correct answer. I would say that it is really the simplicity and uniformity of mathematics that makes it work so well. One real mystery is how the simple objects and rules lead to such incredible complexity.

Since the beginning, mathematicians have been interested in formulating the *foundations* of the subject. What are the simple objects and methods of reasoning in math? In the end, there seems to be a triple of things you need for mathematics: language, logic, and some notion of an entity – i.e., *set theory*. It seems that there is no way to justify *all* the language, logic, and set theory that mathematics uses. The best we can do is make them as simple and as transparent as possible.

If you interpret *language* as “words on a page” then you could easily imagine doing math without language. You might argue that some people “think in pictures”, for instance. But, if you interpret language broadly, as a method of storing and communicating ideas, then it hardly seems possible to make any progress without it.

Once you are able to store ideas and communicate them, the next thing you need is logic – e.g. basic notions of truth and falsehood. Mathematics makes statements to the effect that various things are true and various things are false. Also, you need to know how these words function: For instance, a statement cannot be both true and false. The basic rules of logic are usually formulated as *first order logic*, and then people freely use the rules of first order logic when doing mathematics.

Even after you have language and logic, you need something for them to talk about. Mathematics is primarily a discussion of various kind of entities and their properties – numbers, lines, planes, vector spaces, projective varieties, etc. What are these things? The purpose of these notes is to discuss Zermelo-Frankel (ZF) set theory, a collection of axioms which describes what counts as a mathematical object and how you are allowed to deal with them. However, notice that it is hard to disentangle the three basic components of mathematics. For instance, what are *axioms*? Doesn't a *collection of axioms* about entities seem to already presuppose that there are some entities floating around in advance, namely axioms?

These questions are hard to answer. Maybe the best answer is given by Henri Poincaré about 100 years ago:

Though the source be obscure, still the river flows on.

In writing these notes, I got many of the details about the ZF axioms from Wikipedia and from *An introduction to Set Theory* by William A.R. Weiss, though these notes are not a perfectly faithful account of what is written there.

2 Sets

The foundational axioms of mathematics concern properties of *sets*. Here is an informal definition:

A set is a collection of things, the things themselves being sets.

This is a very beautiful idea: There is only one kind of “thing” (or “object” or “entity” or “structure” or “gadget” or “gizmo”) in the mathematical universe, and it is made of parts which are “things” of the same kind.

On the down side, a set has no formal definition. It is considered to be a basic and undefinable concept. How would you define it? As a collection of things? What are collections? What are things? Any attempt will just lead in circles. Another problem is that the mathematical universe turns out to contain “collections of things” which are not actually sets – e.g., the *class* of all sets. A class is like a set, but bigger. The ZF axioms do not apply to classes, though you sometimes want to refer to classes when making a logical argument which otherwise uses the ZF axioms. Yet another problem is that the axioms assert the *existence* of sets, without saying in what sense sets exists. Probably they don’t exist in the same sense that, say, New Jersey exists. I don’t know any satisfying resolution to these problems, but if you don’t move past them you won’t actually get to mathematics.

Once you grant the existence of sets (whatever they are), you have to swallow the concept of *equality* and *membership*. One writes $A = B$ to indicate that A and B are the same set. One writes $s \in S$ to indicate s is a member of S . That is, s is one of the sets of which S is constituted. Even though the members of a set are themselves sets, they are usually called *members*, or *elements*. Again, ultimately everything (except classes) is a set.

The symbols $\{$ and $\}$ are often used to denote the members of a set. For instance $\{1, 2, 3\}$ denotes the set whose members are 1 and 2 and 3 (whatever they are.) Other useful notions are reducible to these, such as the notion of *subset*. One writes $A \subset B$ if every member of A is also a member of B . That is $a \in A$ implies $a \in B$. Notice that even to talk about the basic definition of a subset, we need a certain amount of language. For instance, we have to know what the word *implies* means. This is what I mean when I say that language and logic are part of the foundations of mathematics just as much as the set-theoretic axioms.

3 The ZF Axioms, Part 1

The ZF Axioms can be divided into two kinds. The first kind are individual statements which do not explicitly rely on first order logic. The second kind is more complicated: it is really an infinite lists of statements whose formulation requires first order logic in an explicit way. The first 4 axioms are pretty straightforward. They correspond to operations in mathematics you are familiar with.

Existence of the Empty Set: There exists a set with no members. It is called \emptyset , the empty set. This axiom is usually considered a consequence of the others, but I think it is a good place to start. This axiom formalizes our notion that *nothing* is an entity in itself.

Axiom of Extension: Two sets are equal if and only if they have the same members. This one seems completely obvious. It formalized our notion of equality.

Axiom of Union: Given a set S , there is a set U_S so that $t \in U_S$ if and only if $t \in A$ for some $A \subset S$. So, U_S is the union of the elements in subsets of S . For instance, if

$$S = \{\{\{1, 2\}, 4\}, \{1, 2\}, \{2, 3\}\}$$

then $U_S = \{\{1, 2\}, 1, 2, 3, 4\}$ and $U_{U_S} = \{1, 2, 3, 4\}$. In compute science, this operation is sometimes called *flattening a list*. This axiom gives us tools to dissect a set, getting at its members, so to speak.

Existence of Power Sets: One set A is a *subset* of a set B if every member of A is also a member of B . For instance $\{1, 2\}$ is a subset of $\{1, 2, 3\}$. Given a set S there exists a set T whose members are the subsets of S . For instance, if $S = \{1, 2, 3\}$, then

$$T = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The power set axiom gives a way to build more complicated sets our of simpler ones.

4 The ZF Axioms, Part 2

Axiom of Pairing: Given sets A and B there exists a set C whose members are A and B . That is $C = \{A, B\}$ exists. One special case is when $A = B$. In this case, the axiom of pairing says that there exists a set $\{A\}$ whose only member is A . This axiom seems rather obscure, but you can think of it as an axiom which goes in the opposite direction of the Axiom of Union. The Axiom of Union has to do with dissecting a set into its components whereas the Axiom of Pairing has to do with building more complicated sets out of simpler ones.

The Axiom of Foundation: Given any nonempty subset S , there exists an element $T \subset S$ such that $T \cap S = \emptyset$. Note that the members of sets are also sets, so this makes sense. The Axiom of Foundation prevents a situation where a set has itself as a member. Suppose that there is a set S such that $S \in S$. By the Axiom of Pairing, there is a set $\{S\}$ whose only member is S . But then the Axiom of Foundation says that there is a member of $\{S\}$ which has no members in common with $\{S\}$. The only member to choose from is S , so S has no members in common with S . But if $S \in S$, then S is a member common to both S and $\{S\}$. This is a contradiction. So, in short, no set can have a member equal to itself.

We've said already that a set is a collection of things, the things themselves being sets. However, we like the idea that the members of a set are somehow simpler than the set itself, so that one can (in principle) start with a set and work backwards, dissecting it into its simpler components, and then dissecting those, etc. You could say that complicated sets are *founded* (or built out of) simpler sets. If some set had itself as a member, this whole enterprise would be derailed. The Axiom of Foundation rules out this bad situation.

Axiom of Infinity There exists a set S which contains the empty set and has the following property. If T is a member of S , then the successor of T is also a member of S . Here, the *successor* of T is defined to be the set $T \cup \{T\}$, the union of T with the set whose only member is T . The Axiom of Infinity guarantees the existence of an infinite set. I will discuss some consequences of this axiom below.

5 First Order Logic

First order logic is a certain collection of *formulas* involving variables, logical connectives, and quantifiers. You should think of a formula as a generalization of the sort of thing you learned about in algebra, like $f(x) = 2x$. Sometimes, *first order logic* is called *the language of set theory*. words like *variable* are parts of our language: A variable is a kind of place-holder for a collection of other words which can be substituted into it. Setting aside what these entities might *be*, first order logic starts from the ground up and defines exactly what counts as a legal formula.

The simplest formulas have the form $A = B$ or $A \in B$. Here A and B are the variables. These are called the *atomic formulas*. Then we have the following rules.

- If S is a formula, so is NOT S .
- If S and T are formulas, so is S AND T .
- One can universally quantify over a formula. Given a formula S , one also has the formula $\forall A S$, which means “ S is true for all A .”

One can make other constructions based on these rules. For instance S OR T is logically equivalent to NOT((NOT S) AND (NOT T)). Likewise $\exists A S$ is equivalent to NOT ($\forall A$ (NOT S)).

So, in short, a *formula* is any collection of symbols which can be built from the atomic formulas using the above rules finitely many times. The formulas acquire truth values when one substitutes sets for all the variables. A formula like $\forall A(A \subset B)$ would say that, for all sets A , the set A is a subset of B . This is clearly false.

Free Variables: Here is a technical notion needed for the last ZF Axiom. Given a formula F and a variable A which occurs in F , you want to minimize the number of times A occurs without changing the (potential) truth value of F . You can do this by replacing all occurrences of $\forall A(..A..)$ and $\exists A(..A..)$ with $\forall \hat{A}(...\hat{A}...)$ and $\exists \hat{A}(...\hat{A}...)$, where \hat{A} is some new variable what does not occur in F . The remaining instances of A are called *free variables*. Call F *clean* if all such replacements have been made.

A *set function* is a clean formula F , with free variables S, x, y_1, \dots, y_n, z , such that for each set x , and each choice of sets y_1, \dots, y_n, S , there is a unique set z such that F is true. One then defines $f(x) = z$.

6 The ZF Axioms, Part 3

Here is the final ZF axiom:

Axiom Schema of Replacement: Given any set S and a set formula f on S as above, there exists a set T such that $z \in T$ if and only if $z = f(x)$ for some $x \in S$. This axiom is saying that, under certain circumstances, we can start with a set S and a function f and *replace* each member $s \in S$ with $f(s)$.

To give an example (which assumes that we're already defined the integers, let $S = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ and let $f(x) = 2x$. This turns out to be a set function. The Axiom Scheme of Replacement guarantees that $\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$ (the even integers) is also a set. To tie this axiom in with something more familiar, you could say that the set S is the *domain* of the function f . The Axiom Schema of Replacement then guarantees the existence of the *range* of f .

The Axiom Schema of Replacement implies another axiom schema which is sometimes also listed as a ZF axiom. Given a formula F with free variables S, x, y_1, \dots, y_n , we get a function f from S into $\{\text{True}, \text{False}\}$ once y_1, \dots, y_n have been specified as sets. The *Axiom Schema of Comprehension* says that there exists a subset $T \subset S$ consisting exactly of those $s \in S$ for which $f(s)$ is true. That is,

$$T = \{s \in S \mid f(s) = \text{True}\}. \quad (1)$$

This is the usual way that one builds new sets out of old ones.

This axiom schema is a consequence of the Axiom Schema of Replacement and the existence of the emptyset. If there is no $s \in S$ for which $f(s)$ is true, we set $T = \emptyset$. Otherwise there is some $s \in S$ for which $f(s)$ is true. In this case we define $g : S \rightarrow S$ by the rule that $g(t) = t$ if $f(s)$ is true and $g(t) = s_0$ if $f(t)$ is false. We can write g as a set-valued function on S and the set given by the Replacement Axiom is as in Equation 1.

There is something a bit disappointing about the Axiom Schema or Replacement and the Axiom Schema of Comprehension. At first you think that the ZF axioms are going to give you a short finite list of *all you need to know* to do math, but then out pop these infinite lists of axioms. On the other hand, there is a sense in which all the axioms on the infinite list are the same.

7 Intersections

The Axiom of Pairing the the Axiom of Union combine to give us a way to take unions of sets in the usual way. I'm not going to talk about the operation of union, but I do want to say something about intersections of sets. This is an operation not explicitly mentioned by the axioms, but which is also a consequence of them.

Suppose that S is a nonempty set and $T \in S$. Then we use the language of set theory and the Axiom Schema of Comprehension to define the intersection of all the members of S which are also members of T . I'll denote this as

$$\bigcap^T S = \{s \in T \mid \forall t \in S, s \in t\}.$$

We think of T as the “container” for the intersection. For example, if

$$S = \{\{1, 2, 3, 4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}, \quad T = \{1, 2, 3, 4\},$$

then

$$\bigcap^T S = \{1\}.$$

8 The Natural Numbers

In the preceding sections, we have been using the symbols $1, 2, 3, \dots$ informally to denote sets, and also as part of the language of set theory. We might have used other symbols like A, B, C, \dots for these purposes, but still you might wonder how the natural numbers are related to the ZF axioms. Here I'll sketch how one defines the natural numbers in terms of the axioms.

Let S be the set from the Axiom of Infinity. Let 2^S denote the set of subsets of S . This set is guaranteed by the Existence of Power Sets. Let $\Sigma \subset 2^S$ denote those elements t such that $\emptyset \in t$ and $u \in T$ implies $u' \in T$. As above u' is the successor of u . In other words Σ consists of all the subsets of S which also satisfy the Axiom of Infinity. The set Σ is guaranteed by the Axiom Schema of Comprehension. Note that $S \in \Sigma$. We define

$$\mathbf{N} = \bigcap^S \Sigma. \tag{2}$$

That is, \mathbf{N} is the smallest set satisfying the Axiom of Infinity.

One can prove that \mathbf{N} consists precisely of \emptyset and the successors of \emptyset . If \mathbf{N} contains any other element s , one can produce a smaller set satisfying the Axiom of Infinity by removing s and all its “predecessors”.

Once \mathbf{N} is defined, we can give the elements of \mathbf{N} their more traditional names.

- 0 is a name for \emptyset .
- 1 is a name for $(0)'$ or $\{\emptyset\}$.
- 2 is a name for $(1)'$, or $\{\emptyset, \{\emptyset\}\}$.
- 3 is a name for $(2)'$, or $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

and so on. This definition has a special beauty: A natural number is a set, with finitely members, each of which is a natural number. Given any two natural numbers, one is a subset of the other. The natural numbers are literally built out of nothing!

With the above definition of natural numbers, you can define $m < n$ if $m \subset n$ and also $n + 1 = n'$. From there, you can derive all the usual rules of arithmetic. Then you can define the integers, the rationals, the reals, complex numbers, matrices, vector spaces, manifolds, etc.

9 The Axiom of Choice

The Axiom of Choice is another axiom one can introduce for sets. Say that a *partition* is a set S whose members are pairwise disjoint. That is, if $A, B \in S$, then $A \cap B = \emptyset$. We insist that the emptyset is not a member of S .

Here is an example. The set

$$\{\{1, 2\}, \{3, 4\}\}$$

is a partition but $\{\{1, 2\}, \{2, 3\}\}$ is not. It is worth pointing that the set S is usually considered to be a partition *of* the union set U_S . For instance, the partition above is a partition of $\{1, 2, 3, 4\}$.

Axiom of Choice: If S is a partition, then there exists a set T such that T has one member in common with each member of S .

For instance, if S is the partition above, we could take $T = \{1, 3\}$.

Sometimes, you can get the result of the Axiom of Choice just using the ZF axioms. For instance, suppose that S is a partition of \mathbf{N} the natural numbers, and each member of S is a finite set of natural numbers. We could then define T to be the set consisting of least elements of members of S . That is, $a \in T$ if and only if a is the least element of the member of S which contains a .

You need to invoke the Axiom of Choice when you do not have a clear selection process like this. For instance, (jumping back to normal mathematical usage for a minute) say that two real numbers are equivalent if and only if their difference is a rational number. The set S of equivalence classes gives a partition of the set \mathbf{R} of real numbers, and here you really need the Axiom of Choice to get the set T .

You might ask: What is the big deal about the Axiom of Choice? Well, let B_r denote the ball of radius r about the origin in \mathbf{R}^3 . Here is a famous theorem.

Theorem 9.1 (Banach-Tarski) *Assume ZF and also that the Axiom of Choice holds for all partitions of \mathbf{R} . Then there are finitely many subsets $S_1, \dots, S_n \subset B_1$ and rigid motions T_1, \dots, T_n of space, so that*

$$B_2 = T_1(S_1) \cup \dots \cup T_n(S_n).$$

That is, you can cut apart the unit ball into finitely many pieces, and rearrange the pieces so that they equal the ball of radius 2.