

1 Complex Numbers

1.1 Algebra of Complex Numbers

The number i is declared by law to satisfy the equation

$$i^2 = -1.$$

A complex number is an expression of the form $x + yi$, with x and y real numbers. Complex numbers are added, subtracted, and multiplied as with polynomials. Examples

$$(2 + 3i) + (5 - 6i) = 7 - 3i.$$

$$(2 + 3i) - (5 - 6i) = -3 + 9i.$$

$$(2 + 3i)(5 - 6i) = 10 + 15i - 12i - 18i^2 = (10 + 18) + i(15 - 12) = 28 + 3i.$$

Division is trickier. Observe that

$$(2 - 3i)(2 + 3i) = 4 + 9 + 6i - 6i = 13.$$

That means that

$$\frac{13}{2 + 3i} = 2 - 3i.$$

Divide both sides by 13:

$$\frac{1}{2 + 3i} = \frac{2 - 3i}{13}.$$

In general,

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2}.$$

Now for division

$$\begin{aligned} \frac{5 - 6i}{2 + 3i} &= (5 - 6i) \times \left(\frac{1}{2 + 3i} \right) = \\ (5 - 6i) \times \frac{2 - 3i}{13} &= \frac{(5 - 6i)(2 - 3i)}{13} = -8/13 - 27/13i. \end{aligned}$$

Problems: Compute $A + B$ and $A - B$ and AB and A/B for

- $A = 2i$ and $B = 1 - i$.
- $A = 1 + 2i$ and $B = 4 - 3i$
- $A = 2 - 5i$ and $B = -2 + 7i$.

1.2 Lengths of Complex Numbers

Let z denote a complex number. The quantity \bar{z} denotes the result of flipping the sign in front of the i coefficient.

$$z = x + yi \implies \bar{z} = x - iy.$$

The “bar” operation is pretty nice. It is called *complex conjugation*. Consider the following example: $z = 2 + 3i$ and $w = 4 + 5i$. Then $\bar{z} = 2 - 3i$ and $\bar{w} = 4 - 5i$ and

$$zw = (2 + 3i)(4 + 5i) = -7 + 22i; \quad (\bar{z})(\bar{w}) = (2 - 3i)(4 - 5i) = -7 - 22i.$$

This one example shows that

$$(\bar{z}) \times (\bar{w}) = \overline{zw}.$$

Problem: Prove that this always works. So, take $z = a + bi$ and $w = c + di$ and calculate the whole thing out.

Here's more notation

$$|z| = \sqrt{z\bar{z}}.$$

In general

$$z = x + yi \implies |z| = \sqrt{(x + iy)(x - iy)} = \sqrt{x^2 + y^2}.$$

Also

$$|z|^2 = z\bar{z} = x^2 + y^2.$$

This looks like the Pythagorean theorem. If you think of z as the point (x, y) in the plane, then $|z|$ measures the distance from z to $(0, 0)$.

The number $|z|$ is called the *norm* of z . The punchline is that

$$|zw| = |z||w|. \tag{1}$$

When you multiply two complex numbers together, their norms multiply. Here is the proof

$$|zw|^2 = (zw)(\overline{zw}) = z \times w \times \bar{z} \times \bar{w} = z \times \bar{z} \times w \times \bar{w} = |z|^2 |w|^2.$$

Take square roots of both sides to get the final result.

1.3 Unit Complex Numbers

A complex number z is called a *unit complex number* if $|z| = 1$. Note that the product of two unit complex number is again a unit complex number, because the lengths multiply and $1 \times 1 = 1$. That is, if $|z| = |w| = 1$ then

$$|zw| = |z||w| = 1 \times 1 = 1.$$

Also,

$$|1/z| \times |z| = |(1/z) \times z| = |1| = 1.$$

So, if z is a unit complex number, then so is $1/z$.

Problem: Show that the following numbers are all unit complex numbers:

$$1 \quad i; \quad \frac{3}{5} + \frac{4}{5}i; \quad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Plot these points in the plane and see what they look like. For instance $i = 0 + 1i$ would be the point with coordinates $(0, 1)$.

Problem: Suppose that z and w are both unit complex numbers. Prove that z/w is also a unit complex number.

The neat thing about unit complex numbers is that you can multiply and divide them and you always get another unit complex number. If you plot all the unit complex numbers in the plane, you get a circle of radius 1. This circle is called the *unit circle*. So, there is a nice way to multiply and divide points on the unit circle.

Problem: Show that the number

$$\left(\frac{s^2 - 1}{s^2 + 1}\right) + \left(\frac{2s}{s^2 + 1}\right)i$$

is a unit complex number for any real number s . This equation gives you one way to generate all the unit complex numbers (except, strangely, for 1, which would require you to take $s = \infty$.)

1.4 Sine and Cosine Functions

The sine and cosine functions are examples of *trigonometric functions*. The word means roughly “derived from triangles”. These functions are unlike any you have seen before because they are not explicitly defined in the same way that polynomials are. There isn’t a “formula” for them in the same way that there is a formula for a polynomial. Well, it turns out that there are formulas, but they are more complicated and I won’t go into it now. The sine and cosine functions (abbreviated \sin and \cos) are defined geometrically in terms of unit complex numbers.

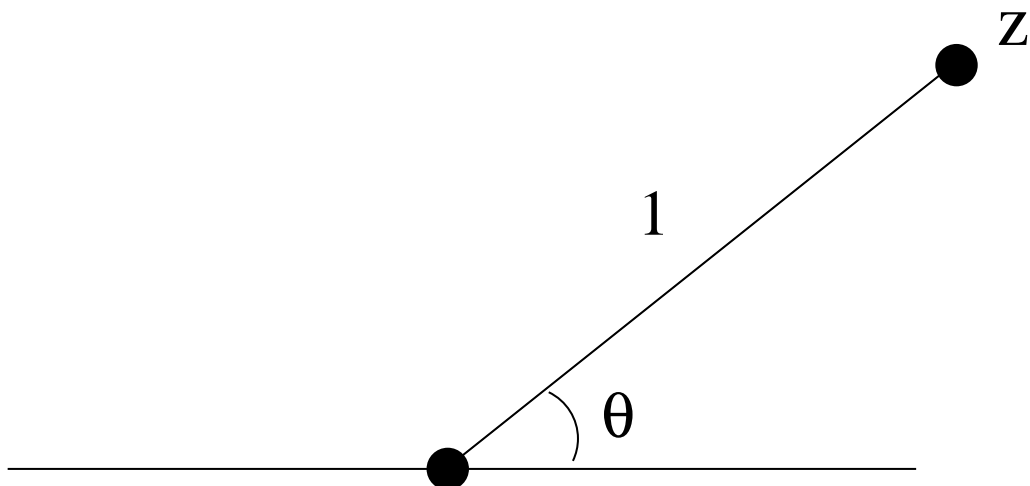


Figure 1: The unit complex number $z = \cos(\theta) + i \sin(\theta)$.

Figure 1 shows a complex number that makes an angle θ with the positive x -axis. The number 1 indicates the length of the edge joining 0 to z .

Depending on θ , the coordinates for z are easy or hard to figure out exactly. In general, the new functions $\cos(\theta)$ and $\sin(\theta)$ are concocted so that the unit complex number

$$z = \cos(\theta) + i \sin(\theta)$$

is the unit complex number that makes an angle θ with the positive x -axis.

Depending on θ , it is sometimes possible to write down an explicit answer to the question *what is $\sin(x)$ and $\cos(x)$* and sometimes not. I’ll give some examples below that show you how to work out some cases. The number of cases that can be worked out like this are pretty small, but still the method will help explain what \sin and \cos mean.

Here are some examples

- When $\theta = 0$ we are looking for the unit complex number that makes an angle of 0 with the x -axis. But the complex number $1 = 1 + 0i$ has this property. So $\cos(0) = 1$ and $\sin(0) = 0$.
- When $\theta = \pi/2$ we are looking for the unit complex number that makes an angle of $\pi/2$ with the x -axis. But π is 180 degrees and $\pi/2$ is 90 degrees. So, we want a unit complex number that is perpendicular to the x -axis. The number $i = 0 + 1i$ has this property. So, $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.
- $\pi/4$ is 45 degrees. The unit complex number that makes an angle of $\pi/4$ with the x -axis is shown in Figure 2. The triangle drawn is a right angled isosceles triangle. So, $z = L + Li$ for the same number L . Since $|z| = 1$ we get

$$|z| = \sqrt{z\bar{z}} = \sqrt{L^2 + L^2} = L\sqrt{2}.$$

So $L = 1/\sqrt{2}$. This means that $\cos(\pi/4) = 1/\sqrt{2}$ and $\sin(\pi/4) = 1/\sqrt{2}$.

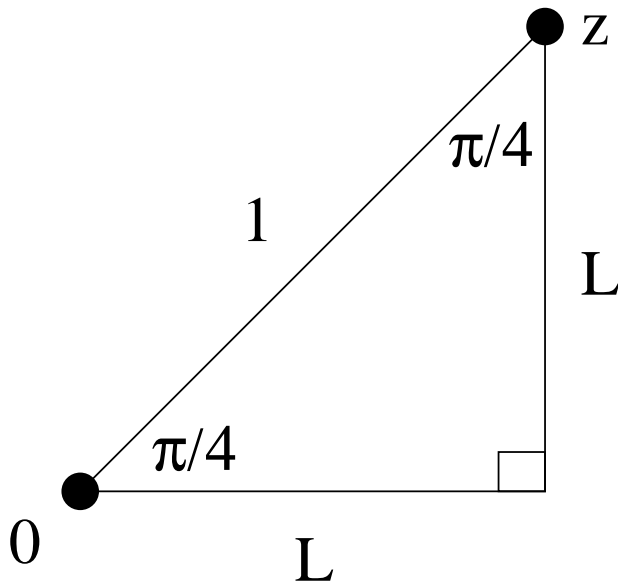


Figure 2: The unit complex number $z = \cos(\pi/4) + i \sin(\pi/4)$.

Challenge Problem: Find $\cos(\pi/3)$ and $\sin(\pi/3)$. Hint: Think about the geometry of the 30 – 60 – 90 triangle.

1.5 Angle Addition

Figure 3 shows the powers of A , when

$$A = \cos(\pi/4) + i \sin(\pi/4) = (1/\sqrt{2}) + (1/\sqrt{2})i.$$

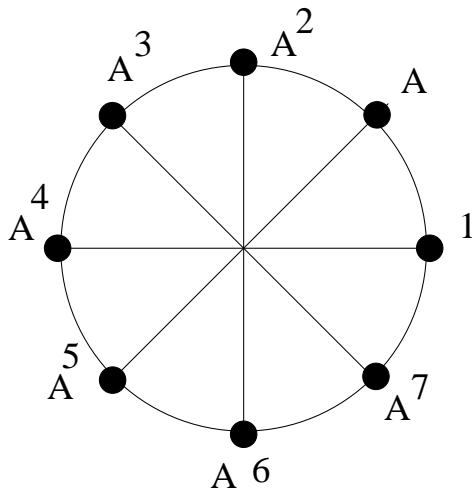


Figure 3: A^k for $k = 1, \dots, 7$.

Note that

$$A^8 = (A^4)(A^4) = (-1)(-1) = 1.$$

Observe also that

$$(A^2)^8 = A^{16} = (A^8)^2 = 1^2 = 1.$$

Similarly

$$(A^3)^8 = A^{24} = (A^8)^3 = 1^3 = 1.$$

And so on. So, the numbers $1, A, \dots, A^7$ are 8 different roots of the equation $x^8 = 1$. These numbers are called *8th roots of unity*.

When you take the n th power of a number, the angle to the x -axis multiplies by n . That is

$$\left(\cos(\theta) + i \sin(\theta) \right)^n = \cos(n\theta) + i \sin(n\theta). \quad (2)$$

Why does it work? The more general statement is that

$$\left(\cos(\theta_1) + i \sin(\theta_1) \right) \times \left(\cos(\theta_2) + i \sin(\theta_2) \right) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2). \quad (3)$$

Equation 3 is called the *angle addition formula*.

Problem: Explain why Equation 3 implies Equation 2.

Problem: Expand out equation 3 and get a formula for $\cos(\theta_1 + \theta_2)$ in terms of $\cos(\theta_1)$ and $\sin(\theta_1)$ and $\cos(\theta_2)$ and $\sin(\theta_2)$.

Proof of the Angle Addition Formula: Let z_1 and z_2 be the two numbers on the left side of Equation 3. Consider the *function* $F(a) = z_1 \times a$. The input of F is a complex number a and the output is another complex number, called $F(a)$. Note 3 things:

1. $F(0) = 0$. So, F doesn't move the origin at all.
2. For any real number $s = s + 0i$, the number $F(s) = sz_1$ is just a scaled copy of z_1 . So, F rotates the x -axis to the line through z_1 .
3. Suppose we have two other points (i.e. complex numbers) a and b . The distance from a to b is $|a - b|$. The distance from $F(a)$ to $F(b)$ is

$$|F(a) - F(b)| = |az_1 - bz_1| = |z_1(a - b)| = |z_1||a - b| = |a - b|.$$

So, F preserves distances between points.

These three properties force F to be a rotation about 0. Since F rotates the x -axis to the line through z_1 , and this line make an angle of θ_1 with the x -axis, we conclude that F is the rotation by θ_1 .

F rotates z_2 to z_1z_2 . Since z_2 makes an angle of θ_2 with the x -axis, the rotated point $z_1z_2 = F(z_2)$ makes an angle of $\theta_1 + \theta_2$ with the x -axis. That's the end of the proof.

Well...almost. There is one irritating point. The three properties above don't quite say that F is a rotation. The other possibility is that F is a *reflection* across some line. Could this really happen? If $z_1 = 1$ then $F(z) = z$ and F is just a rotation by 0 angles. So, when $z_1 = 1$ this doesn't happen. Now imagine continuously changing the choice of z_1 . The reflection option can't suddenly appear out of the blue. That is, the map F can't switch from being a rotation to a reflection as z_1 moves around. Conclusion: The reflection option doesn't happen. That's the end of the proof.

1.6 Geometry of Complex Multiplication

Suppose z is any complex number. Let $r = |z|$. The number z/r is a unit complex number because

$$\left| \frac{z}{r} \right| = \frac{|z|}{|r|} = \frac{|z|}{|z|} = 1.$$

Since z/r is a unit complex number, we can write

$$z/r = \cos(\theta) + i \sin(\theta)$$

for some angle θ . Multiply through by r :

$$z = r(\cos(\theta) + i \sin(\theta)).$$

This way of encoding complex numbers tells the distance from the origin and also the angle that the number makes with the z -axis.

Suppose now that

$$z_1 = r_1(\cos(\theta_1) + i \sin(\theta_1)); \quad z_2 = r_2(\cos(\theta_2) + i \sin(\theta_2)).$$

Combining equations 1 and 3, we get

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right).$$

So, when two complex numbers are multiplied together, their norms multiply and their “angles” add. That’s the geometry of complex multiplication in a nutshell.

Problems: in each case, find a complex number z such that

- $z^4 = -81$.
- $z^6 = 64$.
- $z^6 = 125i$.

Hint: try to find r and θ in the length-angle representation of z .