Math 1040 HW1:

1: Recall that a subset $C \subset \boldsymbol{R}^n$ is *convex* if it has the following property: If $p, q \in C$ are two points then the line segment connecting p to q also belongs to C. Prove that the arbitrary intersection of convex sets is convex. Give an example of a union of 2 convex sets that is not convex.

2: An *isometry* of $\boldsymbol{R}^3$ is a map which preserves distances. A *reflection* of $\boldsymbol{R}^3$ is a map $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ with the following properties:

- $T^2(v) = v$ for all $v \in \boldsymbol{R}^3$.
- There is a plane H in $\boldsymbol{R}^3$ such that $T(h) = h$ for all $h \in H$.
- If $p \in \boldsymbol{R}^3 - H$ then $T(p) \neq p$ and the line segment connecting p to $T(p)$ is perpendicular to H and intersects H at a point equidistant from p and $T(p)$.

Prove that any isometry of $\boldsymbol{R}^3$ is the product of finitely many reflections.

3: (This is exercise 1 in Chapter 8 of the book) Suppose that P is a parallelogram having vertices with integer coordinates. Prove that the area of P is an integer. (Hint: work in $\boldsymbol{C}$ and translate so that the vertices are $0, z, w, z+w$. Then establish the formula that the area of P is the imaginary part of $z\overline{w}$.

4: (This is close to problem 3 in Chapter 8 the book) A *lattice polygon* is one whose vertices have integer coordinates. Suppose P is a lattice polygon which has more than 3 sides. Prove that P can be written as the union of two non-empty lattice polygons which have disjoint interiors.

5: (This is exercise 4 in the book) Prove that scissors congruence is an equivalence relation. This is true for polyhedra of any dimension, but if you like you can just prove it for polygons.

6: Suppose that $C_1, ..., C_n$ are n open convex subsets in the plane. (This means that any point in C_k is contained in a small disk that is also contained in C_k.) Suppose also that every three of these sets has a nontrivial intersection. Prove that they all have a nontrivial intersection. This is known as Helly's Theorem. Hint: First prove it for the case $n = 4$ just to get a feel for it. Then try to make a proof that goes by induction on n.