Graphs and Groups

Rich Schwartz

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1 Definition of a Group

A *group* is a set G together with a binary operation, $*$, which satisfies the following axioms:

- For all $g, h \in G$ the element $g * h$ is defined and lies in G.
- $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.
- There exists a (unique) $e \in G$ such that $g * e = e * g = g$ for all $g \in G$. The element e is called *the identity*.
- For all $g \in G$ there exists a (unique) $h \in G$ such that $g * h = h * g = e$. The element h is called *the inverse of* g and is often written $h = g^{-1}$.

Note: It is not always true in a group that $g * h = h * g$. When this is always true, the group is called *Abelian*.

Here are some examples of groups.

- The group with one element, e. The rule is $e * e = e$.
- G is any of $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ and $*$ is addition. Here $e = 0$ and g^{-1} is usually written $-g$.
- G is any of $\boldsymbol{Q} - \{0\}$, $\boldsymbol{R} - \{0\}$ or $\boldsymbol{C} - \{0\}$ and $*$ is multiplication. Here $e = 1$ and g^{-1} is usually written $1/g$.
- $\boldsymbol{Z}/n$, the group of residue classes of integers mod n. Here $*$ is addition mod n.

- Let X be any set and let S_X denote the set of bijections from X to itself. In this case $*$ is composition, and e is the identity map, and g^{-1} is the inverse of g in the sense of mappings. When $X = \{1, ..., n\}$ the group S_X is often denoted S_n and called *the permutation group*.

- Whenever G_1 and G_2 are groups, the product $G_1 \times G_2$ is a group in a natural way. The two group laws are done coordinatewise.

- The set of rigid motions of Euclidean space forms a group. The group law is composition. This group is known as the *Euclidean group*.

- The same goes for the group of rotations of the sphere. This group is called the *rotation group*, at least when the dimension of the sphere is specified in advance.

A *subgroup* H of a group G is a subset which is also a group, with respect to the same operation. Here are some examples of subgroups.

- The trivial group is a subgroup of every group.

- The set of powers of a single element $g \in G$, namely

$$g, g * g, g * g * g, ...$$

  is a subgroup of any finite group. Such a group is known as a *cyclic subgroup*. It is not unlike a cycle in a graph.

- The set $2\boldsymbol{Z}$ of even integers is a subgroup of $\boldsymbol{Z}$.

- When $X \subset Y$, there is a natural way to think of S_X as a subgroup of S_Y.

- Each element g of S_n can be considered as a linear transformation $T_g : \boldsymbol{R}^n \to \boldsymbol{R}^n$. The map T_g sends the standard basis element e_j to the basis element $e_{g(j)}$. That is,

$$T_g\Big(\sum a_j e_j\Big) = \sum a_j e_{g(j)}.$$

  It turns out that $\det(T_g) = \pm 1$. The element g is called *even* if $\det(T_g) = 1$. The set of even permutations of S_n is a subgroup. It is denoted by A_n. It has $n!/2$ elements.

2 Group Isomorphisms

Here is a little more information about group theory. An *isomorphism* between groups G_1 and G_2 is a bijection $f : G_1 \to G_2$ which respects the group laws. That is,

$$f(a * b) = f(a) * f(b), \qquad \forall\ a, b \in G_1.$$

Here the first instance of $*$ is the group law in G_1 and the second instance is the group law in G_2. When two groups are isomorphic, they are essentially the same group, except that the elements have different names.

Here is a nice example. Let G_1 be the group of nonzero real numbers, with respect to multiplication. Let G_2 be the real numbers with respect to addition. Let $f(x) = \log(x)$. The famous equation

$$\log(ab) = \log(a) + \log(b)$$

really says that the log function is an isomorphism between the two groups.

Theorem 2.1 (Cayley) *Let G be a group. G is isomorphic to a subgroup of S_G.*

Proof: Given an element $g \in G$ we want to cook up a permutation $T = f(g)$ of G (as a set). Here is the formula

$$T(h) = g * h.$$

The map T is a bijection because the inverse map is given by

$$T^{-1}(h) = g^{-1} * h.$$

The associative law shows that

$$f(g) \circ f(h) = f(g * h).$$

Hence, our map $f : G \to S_G$ is an isomorphism onto the image $f(G)$, which is a subgroup. ♠

So, if you don't like abstract groups, you can think of a group as a subgroup of the group of bijections of some set. This is often not the best way to think of a group, but sometimes it is useful.

3 Automorphisms of Graphs

Let Γ be a graph. An *automorphism* of Γ is a graph isomorphism from Γ to itself. In other words, an automorphism of Γ is a permutation of the vertices of Γ which maps incident vertices to incident vertices. If you take two automorphisms of Γ, you can compose them and get a third. Thus, the set of automorphisms of Γ forms a group, denoted $A(\Gamma)$. The identity element is the automorphism which maps every vertex to itself. The inverse of an automorphism is just the inverse permutation.

Note that $A(\Gamma)$ automatically is defined as a subgroup of a permutation group – though if you were to run Cayley's Theorem you'd get a different permutaton group! To be even more concrete, suppose Γ is a finite graph. If you label the vertices of Γ as $1, ..., n$ then every element of $A(\Gamma)$ is naturally an element of S_n. That is $A(\Gamma) \subset S_n$. (Note that Cayley's theorem would give $A(\Gamma) \subset S_N$ where N is the order of $A(\Gamma)$. Often $N \neq n$.) Here are some examples.

- The automorphism group of the graph with one vertex is the trivial group.
- The automorphism group of K_n is S_n.
- The automorphism group of any nontrivial path is $\boldsymbol{Z}/2$. All you can do is reverse the direction of the path, so to speak.
- The automorphism group of the cycle C_n has $2n$ elements. Half the elements are cyclic permutations of the vertices and the other half are "reflections". This group is often denoted D_n, and it is an example of a *dihedral group*.
- The automorphism group of the Peterson graph is S_5. Here is a nice way to think about it. You can make a graph whose vertices are 2 element subsets of $\{1, 2, 3, 4, 5\}$ and whose edges join disjoint subsets. For instance $\{1, 2\}$ is joined to each of $\{3, 4\}$ and $\{4, 5\}$ and $\{3, 5\}$. This is a graph with 10 vertices in which every vertex has degree 3. If you draw it, you will see that it is the Peterson graph. Any permutation of $\{1, 2, 3, 4, 5\}$ naturally gives a permutation of the graph.
- Make the graph whose edges are the seams on a soccer ball. The group of automorphisms of the graph which can be achieved by rotations of the soccer ball is A_5.

4 Cayley Graphs

Let G be a group. A *generating set* for G is a subset $S \subset G$ such that every nontrivial element of G is a product of elements of S. That is

$$g = s_1 * ... * s_n.$$

Here n depends on g and so to $s_1, ..., s_n \in S$. Here are some examples.

- $G = \boldsymbol{Z}/n$ and $S = \{1\}$.
- $G = \boldsymbol{Z}$ and $S = \{\pm 1\}$.
- $G = \boldsymbol{Z}$ and $S = \{\pm 2, \pm 3\}$. This also works when S is any pair of relatively prime integers (and their negatives).
- $G = S_n$ and S is the set of transpositions. A *transposition* is a permutation which switches two elements and fixes the others.
- $G = A_5$ and S is the two permutations

  $$A = (12345), \qquad B = (12)(34)(5).$$

  This is the cycle notation for permutations. The permutation A cycles the indices by adding 1 mod 5. The permutation B swaps 1 and 2, swaps 3 and 4, and fixes 5. Note that these are both even permutations. It is a bit of work to show that $\{A, B\}$ generates A_5.

The *Cayley graph* $\Gamma(G, S)$ is a directed graph whose vertices are the elements of G and whose directed edges have the form

$$g \to g * s, \qquad \forall g \in G, \quad \forall s \in S.$$

When G is a finite group and $|S| = k$, every vertex of G has detree $2k$. However, there is a convention that when $s \in S$ has order 2, the two directed edges

$$g \to g * s, \qquad g * s \to g * s * s = g,$$

are collapsed into a single undirected edge. So, in the last example given above, the Cayley graph $\Gamma(A_5, \{A, B\})$ would have degree 3 with this convention.

Let's work out the Cayley graphs of the examples given above.

- $\Gamma(Z/n, \{1\})$ is C_n.
- $\Gamma(\boldsymbol{Z}, \{\pm 1\})$ is the bi-infinite path.
- It is most convenient to draw $\Gamma(\boldsymbol{Z}, \{\pm 2, \pm 3\})$ on an infinite grid, as suggested by Figure 1.

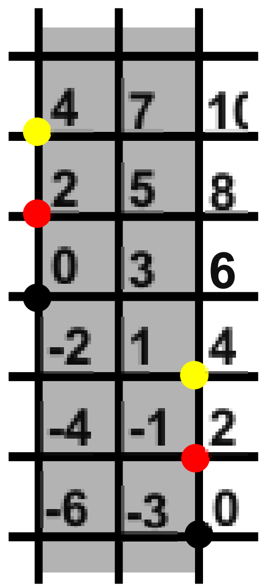

Figure 1: Numbers on a grid

To get the graph, cut out the infinite grey strip and glue the sides according to the rule suggested by the colored vertices. This gives you a kind of grid on an infinite cylinder. You can check that each integer appears exactly once on this cylindrical grid. Were you to use other relatively prime pairs of integers, you would get a similar kind of cylindrical grid, though typically the cylinder would be fatter.

- This next example is hard to draw in general, but Figure 2 shows what you get for the case $n = 3$. In this case S_3 has 6 elements. Using the convention above, Γ has degree 3. The edges are color-coded according to which element in S they represent.

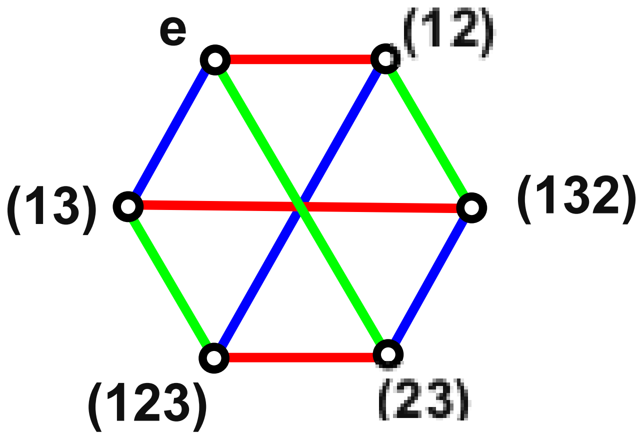

Figure 2: Numbers on a grid

- For the last example, the graph has 60 vertices and (with the convention) each vertex has degree 3. Note

$$A * A * A * A * A = e, \qquad A * B * A * B * A * B = e.$$

So, every vertex of the Cayley graph belongs to two 6-cycles and one 5-cycle, as indicated in Figure 3.

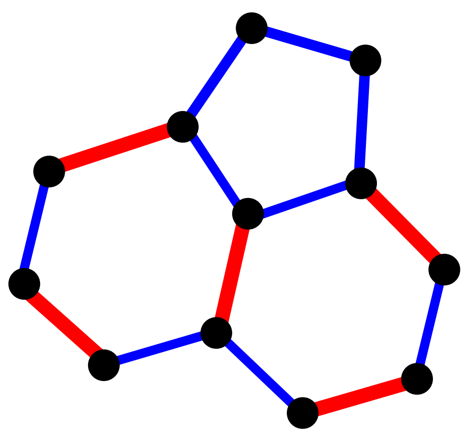

Figure 3: Part of a soccer ball

Each vertex is contained in a unique 5-cycle. Since there are 60 vertices, there are 12 5-cycles. Each of these 5-cycles is surrounded by 6-cycles. At the same time, each 6-cycle shares an edge with 3 5-cycles and 3 6-cycles. The only possible pattern is that of the soccer ball graph.

5 Quasi-Isometries

The Cayley graph $\Gamma(G,S)$ depends a lot on the choice of S. For instance, $\Gamma(G,G)$ is always the complete graph on $|G|$ vertices, at least if we follow the convention of collapsing directed 2-cycles into single edges. (This is a generalization of the convention mentioned above.) It turns out that the situation is a bit different for $\Gamma(G,S)$ when G is an infinite group and S is a finite set. In a sense, the Cayley graph does not depend much on S. This section will explain the sense in which this is meant.

Let X and Y be *metric s[aces*. Given some $K \geq 1$, a K-*quasi-isometry* is a map $f : X \to Y$ with two properties. First,

$$(1/K)d_X(a,b) - K \leq d_Y(f(a), f(b)) \leq K d_X(a,b) + K, \qquad \forall a, b \in X.$$

Second, for all $b \in Y$ there is some $a \in X$ such that $d_Y(b, f(a)) \leq K$. These two conditions say that f is more or less a bijection which does not distort distances two badly.

Two spaces X and Y are said to be *quasi-isometric* if there is a K-quasi-isometry between them for some K. It turns out that this is an equivalence relation. So, is natural to consider metric spaces up to the equivalence of quasi-isometry. Here are some examples.

- Every compact metric space is quasi-isometric to a point. So, this equivalence relation is pretty boring for compact spaces.

- $\boldsymbol{Z}^n$ and $\boldsymbol{R}^n$ are quasi-isometric. You can take $K = \sqrt{n} + 1$. Allowing the additive slop $\pm K$ in the definition allows two spaces to be quasi-isometric even when they look very different on small scales. When we look at tables and chairs and walls we usually think of them as continuous objects, but they are mostly empty space and very granular on small scales. So, our brains naturally do quasi-isometries.

- If K is any compact metric space and X is an arbitrary metric space, then X and $X \times K$ are quasi-isometric. For instance, the line and the infinite cylinder are quasi-isometric. This is also something our brain does. When we look at a telephone wire we might think of a line, but really it is a kind of tube. We are willing to forget a bounded amount of thickness.

- It turns out that the line and the ray are not quasi-isometric.

- It turns out that $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ are quasi-isometric if and only if $m = n$.
- When $k, \ell \geq 3$, the regular infinite tree of degree k is quasi-isometric to the regular infinite tree of degree ℓ. This is one of your HW problems in the case $k = 3$ and $\ell = 4$.

Any graph is naturally a metric space. We think of each edge as being a copy of the unit line segment, and then the distance between any two points in the graph is the length of the shortest path. For instance, the distance between adjacent vertices is 1, and the distance between the midpoint of an edge and either incident vertex is 1/2. The graph is naturally a union of unit length segments.

Theorem 5.1 *Let G be an infinite group and let S, T be two finite generating sets. Then $\Gamma(G, S)$ and $\Gamma(G, T)$ are quasi-isometric.*

Proof: Let $\Gamma_S = \Gamma(G, S)$ and likewise define Γ_T. Let $f : \Gamma_S \to \Gamma_T$ be the identity on vertices, and arrange that f maps each edge to one of the two vertices incident to it. The choice doesn't matter. It suffices to prove that f is a quasi-isometry when restricted to the vertices, because every edge is at most 1 unit away from its endpoints.

Let $s_1, ..., s_m$ be the elements of S and let $t_1, ..., t_n$ be the elements of T. The distance between vertices a, b in Γ_S is the minimum number k such that $b^{-1}a = s_1 * ... * s_k$, for $s_1, ..., s_k \in S$. A similar statement applies to Γ_T. There is some K such that

- Each element of S can be written as the product of at most K elements of T.
- Each element of T can be written as the product of at most K elements of S.

This means that distances between vertices, relative to the two graphs, are the same up to a factor of K. Hence, on vertices, the map f is a K-quasi-isometry. As mentioned above, this is all we need to know. ♠