Brooks's Theorem

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February 13, 2020

These notes prove the following theorem.

Theorem 0.1 (Brooks) *Let G be a graph having maximum degree n. Suppose also that G is not an odd cycle or the complete graph K_n. Then G can be properly colored using at most n colors.*

Note that an odd cycle has max degree 2 but requires 3 colors, and K_{n+1}, which has max degree n, requires $n+1$ colors. So, the conditions in the theorem are necessary.

The proof here has a strategy similar to what is in §5.1 of West's book, but my argument avoids bringing in block decompositions. I think that this argument is easier.

0.1 The Non-Regular Case

Suppose G has a vertex v with degree less than n. Let N be the number of vertices of G. We set $v_N = v$. Let T be any spanning tree for G. We can order the vertices so that their indices increase along any path of T that leads to v_N. This means that for each vertex $v_i \neq v_N$ there is some other vertex v_j incident to v_i and having $j > i$.

Assuming that we have chosen colors $C_1, ..., C_i$ for vertices $v_1, ..., v_i$, and $i < N$, we note that we have <u>not</u> colored at least one vertex incident to v_i, namely our vertex v_j mentioned above. That is, v_i is incident to at most $n-1$ already-colored vertices. So, we may choose a color for v_i which does not conflict with any previous choices. This works all the way until we get to v_N. But v_N is, by definition, incident to at most $n-1$ vertices. So, we can color v_n in such a way that there are no conflicts.

0.2 Nice Graph Case

We call G a *nice graph* if G has 3 vertices v_1, v_2, v_N such that

- v_1 and v_2 are both incident to v_N but not incident to each other.
- $G - v_1 - v_2$ is connected.

Now we show that Brook's Theorem holds for nice graphs. We let T be a spanning tree for $G - v_1 - v_2$ and we use T to order the points of G as $v_1, v_2, v_3, ..., v_N$ where each vertex v_i for $i \geq 3$ is incident to at least one vertex v_j with $j > i$.

We start by coloring v_1 and v_2 both blue – this does not introduce a conflict because these vertices are not incident to each other. We now proceed as above. We can use our n-colors to color $v_3, ..., v_{N-1}$ with no conflicts. Finally, consider v_N. Since v_N has degree n and is incident to two vertices having the same color, there is still a color left over for v_N. So, we can complete the n-coloring.

0.3 The Cut Vertex Case

From now on we can assume that G is regular and has degree $n \geq 3$. Suppose that G has a cut vertex – i.e. a vertex v such that $G - v$ is disconnected. Let $G_1, ..., G_m$ be the components of $G - v$. Let $\widehat{G}_k$ denote the graph obtained by starting with G_k and adding back v and all the edges of v which connect to vertices of G_k. The graph $\widehat{G}_k$ has max degree n and is not regular. So, by the non-regular case, we can n-color the vertices of $\widehat{G}_k$. Permuting the colors, we can assume that v is blue in $\widehat{G}_k$ for all k. But each $\widehat{G}_k$ is a subgraph of G, and the only edges between $\widehat{G}_i$ and $\widehat{G}_j$ for $i \neq j$ involve v. So, the individual colorings piece together to give an n-coloring of G.

0.4 The Remaining Case

Suppose that G is n-regular for $n \geq 3$, and not nice. Since G is not the complete graph, there are a pair of vertices v, w, not incident to each other but both incident to some third vertex. Since G is not nice, $G - v - w$ is disconnected. Let $G_1, ..., G_m$ be the components of $G - v - w$. Since G has no cut vertex, v and w are both incident to vertices in G_k for each k.

Let $\widehat{G}_k$ be the graph obtained by adding v and w back to G_k, then adding back all edges between v, w and G_k. Let e be some extra edge joining v to

w. Note that the graph $\widehat{G}_k \cup e$ has max degree n and fewer vertices than G. Note also that $\widehat{G}_k \cup e$ is not a cycle because some vertex of $\widehat{G}_k \cup e$ has degree $n > 2$. There are 2 cases to consider.

Case 1: Suppose that no $\widehat{G}_k \cup e$ is a complete graph. Then by induction on the number of vertices, we can n-color each of these graphs. The vertices v and w get distinct colorings because they are adjecent in $\widehat{G}_k \cup e$. Thus we can n-color $\widehat{G}_k$ in such a way that v and w get distinct colors. We can permute the colors so that the two common vertices v, w get the same two colors in all cases. We then piece the colorings together just as in the cut vertex case.

Case 2: Suppose that (after renumbering) $\widehat{G}_1 \cup e$ is the complete graph. The degree of v in $\widehat{G}_k \cup e$ is n. Hence v is incident to $n-1$ edges in G_1. But v is also incident to at least one edge in each of $G_2, ..., G_m$. Since the degree of v is n, we see that in fact $m = 2$ and only one edge connects v to a vertex v' of G_2. Likewise w is incident to just one vertex w' of G_2. It could happen that $v' = w'$. This doesn't matter.

- The graph $\widehat{G}_1$ is the complete graph K_n minus a single edge, and hence non-regular. So, we can n-color $\widehat{G}_1$. Both v and w must get the same color, because we cannot n-color $\widehat{G}_1 \cup e$. We permute the colors so that v and w are colored blue.

- Since v' has degree less than n, we see that G_2 is not regular. We can therefore n-color G_2. We can permute the colors of G_2 so that v', w' are not blue. But then we can extend our coloring to $\widehat{G}_2$ so that both v and w are blue.

$\widehat{G}_1$ and $\widehat{G}_2$ are n-colored in such a way that in both graphs v and w are blue. Now we piece together the colorings just as above.