The Countable Hall Matching Theorem

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Suppose that A and B are both copies of $\boldsymbol{N}$, the set of natural numbers, and G is a bipartite graph having A and B as the two vertex sets. Suppose that G has 3 properties:

1. For each finite subset $S \subset A$, we have $|N(S)| \geq |S|$.
2. For each finite subset $S \subset B$, we have $|N(S)| \geq |S|$.
3. Every vertex of G has finite degree.

Here $N(S)$ is the set of neighbors of S in the graph. Under these conditions, I'll prove that G has a perfect matching. (The third condition is probably not needed, but I use it in the proof.) A nice corollary of this result is that any countably infinite regular bipartite graph has a perfect matching. The result here will probably also be useful on the homework assignment.

Proof Outline: I'll show that Condition 1 implies that there is a matching α on G which saturates A. That is, every vertex of A is matched. The same argument, with A and B switched, shows that there is a matching β on G which saturates B. We can think of α as an injective map from A to B and β as an injective map from B to A. By the Cantor-Bernstein Theorem, there is a bijective map h from A to B. Examining the proof of the Cantor-Bernstein Theorem, we see that the bijection h either coincides with α or with β^{-1} at each vertex of A. Therefore the vertices a and $h(a)$ are always connected by an edge of G. So, h defines a perfect matching on G. To finish the overall proof, we just have to show that Condition 1 implies the existence of a matching that saturates A.

Main Step: Now I'll show that Condition 1 implies that there is a matching α on G which saturates A. Let G_n be the graph obtained as follows: Let A_n be the first n vertices of A and let B_n be all the neighbors of all the vertices of A_n. Then G_n consists of A_n, B_n, and all the edges of G connecting vertices of A_n to vertices of B_n. We have $G_1 \subset G_2 \subset G_3 ....$ These are all finite graphs because every vertex has finite degree.

We say that a subset $S \subset A$ is *k-coherent* if all the matchings corresponding to indices in S do the same thing to all the vertices $1, ..., k \in A$. Since G has finite degree, there are only finitely many possibilities for what one of our matchings can do to the set $\{1, ..., k\}$. Hence there is an infinite k-coherent set. Indeed, every infinite subset of $\boldsymbol{N}$ contains a k-coherent set. So, working inductively, we construct infinite coherent subsets $S_1 \supset S_2 \supset S_3 ...$ It may happen that the intersection of these sets is empty, but we don't care.

We define α as follows: The restriction of α to $\{1, ..., k\}$ equals the restriction of α_n to $\{1, ..., k\}$ for any $n \in S_k$. This consistently defines α because our sets are nested: $S_{k+1} \subset S_k$. By construction, α is a matching on G which saturates A. That's the end of the proof.