Notes on Sperner's Lemma

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1 Sperner's Lemma

Sperner's Lemma applies to any cleanly triangulated triangle. An example is shown on the left hand side of Figure 1. The drawing on the right is not meant to be an example; it shows something we want to avoid. The main things you want from the triangulation are that the small triangles meet edge to edge and pairwise have disjoint interiors. We call this a *clean triangulation*.

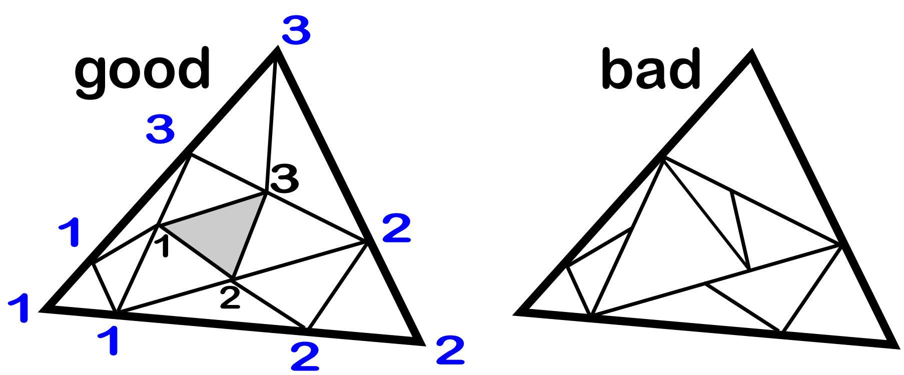

Figure 1: A good triangulation and a bad one in action.

Let T be the triangle, and suppose it is cleanly triangulated into n smaller triangles $T_1, ..., T_n$. Suppose also that the vertices of the triangulation are labeled by integers $\{1, 2, 3\}$ so that the kth side of T has no k label. Sperner's Lemma says that some T_j gets all 3 labels. This result generalizes to n dimensions in a way which I will leave to you.

First Proof: Here is the reference for the amazing proof to follow.

A. McLennan and R. Tourkey, *Using volume to prove Sperner's lemma* Econ. Theory **35** (2008) pp 593-597

We're going to assume that we have a labeling in which no triangle sees all three labels and derive a contradiction. We normalize so that T has area 1. Let's call the vertices of the triangulation $V_1, ..., V_m$. Let $L_1, ..., L_m$ be the labels of these vertices. Let $W(1)$, $W(2)$, $W(3)$ be the vertices of the big triangle T. For each $t \in [0,1]$, define the new point

$$V_k(t) = (1-t)V_k + tW(L_k). \tag{1}$$

In other words, we think of t as time and we think of a the curve $t \to V_k(t)$ as a path which starts at V_k and ends at $W(L_k)$ and moves in a straight line at constant speed. Supposing that the triangle T_k has vertices A, B, C, let $T_k(t)$ be the triangle with vertices $A(t), B(t), C(t)$. As the points move with t these triangles change shape.

Consider the function

$$f(t) = \sum_{j=1}^{n} \text{area}(T_j(t)). \tag{2}$$

First of all, this function is a polynomial, thanks to the simple formulas – e.g. determinants – one can use to compute the areas of the triangles. Second of all, $f(t) = 1$ for all t sufficiently close to 0. The point is that, for small t, we still have a triangulation even though the points have moved a little. Since f is a polynomial, we must have $f(t) = 1$ for all $t \in [0,1]$. However, suppose that no triangle T_k sees all three labels. Then $T_k(t)$ converges either a single vertex or to an edge of T as $t \to 1$. Hence

$$\lim_{t\to 1} \text{area}((T_k(t)) = 0. \tag{3}$$

Since this is true for all triangles, and there are only finitely many of them in the triangulation, we see that $f(t) \to 0$ as $t \to 1$. This is a contradiction. That's the end of the proof.

Second Proof: This is the more traditional proof. Say that a *flag* is a pair (T_k, e), where T_k is one of the triangles of the triangulation and e is an edge of T_k. Each triangle participates in 3 flags and each edge either participates in 1 or 2 flags, depending on whether the edge is in the boundary of T. Say that (12)-flag is a flag which has the labels 1 and 2 on its edge. We're going to count the (12)-flags in two ways.

Let's first count these flags going edge to edge. Each interior edge contributes an even number of (12)-flags to the total, because it participates in two flags and these two flags are either simultaneously (12)-flags or not (12)-flags. The only boundary edges which contribute a (12)-flag are the ones on the side which has the 1 and 2 labels. This side is divided into finitely many edges. One endpoint of the side is labeled 1 and the other one is labeled 2. So, as we go from one endpoint to the other, we have to switch labels an odd number of times. Hence, there are an odd number of (12)-flags coming from the boundary edges. But that means there is an odd number of (12)-flags overall.

On the other hand, let's count the (12)-flags triangle by triangle. As I mentioned above, each triangle participates in 3 flags. Just list out the possibilities and you can see that a triangle contributes an odd number of (12)-flags to the count if and only if it gets all 3 labels. The triangles labeled $(1,2,2)$ $(1,1,2)$ each contribute two (12) flags to the count and the rest contribute zero. Hence, there must be an odd number of triangles which are labeled $(1,2,3)$. That completes the proof.

2 A Topological Interlude

Before going further, we need a foundational result from topology. In class we proved the following result: If Δ is a (closed, solid) triangle and $\{a_n\}$ is an infinite sequence of points in Δ, then there is some point $a \in \Delta$ such that some subsequence converges to a_n. Really, we proved this when Δ was the unit square, using divide and conquer, but it works just the same for a triangle.

Lemma 2.1 (Uniform Continuity) *Suppose $f : \Delta \to \boldsymbol{R}$ is a continuous function and $\epsilon > 0$ is given. Then there exists some $\delta > 0$ with the following property. For any two points $a, b \in \Delta$ with $d(a,b) < \delta$, we have $|f(a) - f(b)| < \epsilon$.*

Proof: Suppose this is false. Then we can find sequences $\{a_n\}$ and $\{b_n\}$ such that $d(a_n, b_n) \to 0$ but $|f(a_n) - f(b_n)| \geq \epsilon$. By the result above, there is a subsequence of $\{a_n\}$ which converges to a point $a \in \Delta$. Passing to a subsequence, we might as well assume that $a_n \to a$. But then $b_n \to a$ as well. Since f is continuous at a, there is some δ such that $|f(p) - f(a)| < \epsilon/2$ once $d(p, a) < \delta$. But than $|f(a_n) - f(b_n)| < \epsilon$ once n is large enough for both these points to be within δ of a. This contradiction proves the lemma. ♠

If you know what a compact metric space is, then you'll probably know that the same result works for any compact metric space.

3 Applications

Suppose that Δ and Y are two metric spaces, with $Y \subset \Delta$. A *retraction* from Δ to Y is a continuous map $f : \Delta \to Y$ such that f is the identity map on Y. In the next result, Δ will be a triangle and Y will be the boundary of Δ – i.e. just the edges of the triangle.

Theorem 3.1 *There is no continuous retraction from a triangle onto its boundary.*

Proof: It suffices to consider the case when Δ is an equilateral triangle having side length 100.

We will suppose that f is a retraction and derive a contradiction. By the Uniform Continuity Lemma, we can find some δ so that $|f(a) - f(b)| < 1$ for all a, b in the triangle with $|a - b| < \delta$. Choose a triangulation of Δ where each triangle in the triangulation has side length less than δ.

Label a vertex v of the triangulation by the name of the vertex of Δ closest to $f(k)$. In case of a tie, choose the lower label. f maps the vertices of each small triangle so that they are each within 1 of each other. But then no triangle can be labeled $(1, 2, 3)$ because the three vertices of Δ are spread far apart. On the boundary, the labeling is more or less as in Figure 1. So, this labeling contradicts Sperner's Lemma. ♠

Here's a very similar formulation.

Theorem 3.2 *There is no retraction from a disk onto its boundary.*

Proof: Let Δ be an equilateral triangle and let D be a disk. Let ∂D and $\partial\Delta$ denote the boundaries of D and Δ.

There is a fairly obvious continuous map $h : \Delta \to \boldsymbol{D}$, illustrated by Figure 2, with the property that h is a bijection and h^{-1} is also continuous. Such a map is called a *homeomorphisms*. In Figure 2, you just stretch each radial line segment so that instead of going from the center $\partial\Delta$, it goes from the center to ∂D.

Note that h is also a homeomorphism from $\partial\Delta$ to ∂D. If $f : D \to \partial D$ was a retraction then $h^{-1}fh$ would be a retraction from Δ onto its boundary. This is impossible. ♠

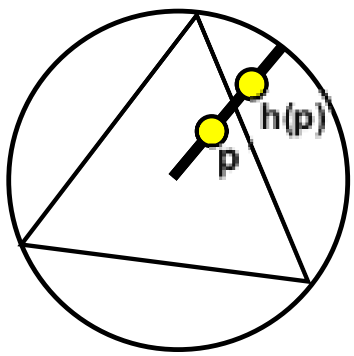

Figure 2: Homeomorphism from Δ to D.

Here is the main application. This is the famous Brouwer fixed point theorem.

Theorem 3.3 *Suppose that D is a disk and $f : D \to D$ is a continuous map. Then f has a fixed point. That is, there is some $p \in D$ such that $f(p) = p$.*

Proof: We will assume that f has no fixed points and derive a contradiction. Given any $p \in D$. consider the ray R_p which starts at $f(p)$ and moves in the direction of p. Let $g(p) = R_p \cap \partial D$, the point where the ray intersects the boundary. It is easy to check that the function g is continuous. If you move p a little, then the ray does not change much, either in terms of slope or starting point. So, it hits ∂D at a nearby point. At the same time, $g(p) = p$ when $p \in \partial D$. Hence g is a retraction. Contradiction. ♠