Notes on Matrices: The purpose of these notes is to give some background in linear algebra that pertains to the Matrix Tree Theorem. The main results presented here are solutions to problems 8.6.18 and 8.6.19.

Circulent Matrices: Given a vector $A = (a_0, ..., a_{n-1})$, define the square matrix

$$M_A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \dots \end{bmatrix} \tag{1}$$

It is easy to compute the eigenvalues and eigenvectors of A. Let

$$\omega = \exp(2\pi i/n) \tag{2}$$

be the usual nth root of unity. Define

$$V_k = (1, \omega^k, \omega^{2k}, \omega^{3k}, ...), \qquad \lambda_k = V_k \cdot A. \tag{3}$$

We compute

$$M_A(V_k) = \begin{pmatrix} a_0\omega^{0k} + a_1\omega^{1k} + ... + a_{n-1}\omega^{(n-1)k} \\ a_{n-1}\omega^{0k} + a_0\omega^{1k} + ... + a_{n-2}\omega^{(n-1)k} \\ \dots \end{pmatrix}$$

In short

$$M_k(V_k) = \lambda_k V_k. \tag{4}$$

The vectors $\{V_0, ..., V_{n-1}\}$ form an eigenbasis for M_A. The corresponding eigenvalues are $\lambda_0, ..., \lambda_{n-1}$. In particular

$$\det(M_A) = \prod_{k=0}^{n-1} \lambda_k. \tag{5}$$

One nice example is the case of the matrix M_A, where $A = (n, -1, ..., -1)$. The total length of A is n. In this case, we have $\lambda_0 = 1$ and, for $k > 0$,

$$\lambda_k = n - \sum_{i=1}^{n-1} \omega^{ki} = (n+1) - \sum_{i=0}^{n-1} \omega^{ki} = n + 1. \tag{6}$$

Therefore

$$\det(M_A) = (n+1)^{n-1}. \tag{7}$$

Cofactors of Special Matrices: Given any $n \times n$ matrix M, we have the *cofactor*

$$C_{ij} = (-1)^{i+j} \det(M^*_{ij}), \tag{8}$$

where M^*_{ij} is the $(n-1) \times (n-1)$ minor obtained by crossing out the ith row and the jth column of M. Let M^* denote the matrix whose (ij)th entry is C_{ji}. That is M^* is the transpose of the matrix of cofactors. We have the familiar formula

$$MM^* = (\det M)I_n, \tag{9}$$

where I_n is the identity matrix.

Say that M is *special* if M is symmetric and the sum of entries in every row of M is zero. Since M is symmetric, this means that the sum of entries in every column of M is also 0. Note that $\det(M) = 0$ when M is a special matrix. The point is that $M(1, ..., 1) = 0$, so that M has a kernel. Say that a special matrix is *nice* if its kernel is precisely the span of $(1, ..., 1)$. We can identify the set of special matrices as a linear subspace of $\boldsymbol{R}^{n^2}$, and so it makes sense to talk about metrical concepts such as density.

Lemma 0.1 *The set of nice special matrices is dense in the set of all special matrices.*

Proof: Let X denote the space of special matrices and let $S \subset X$ denote the set of nice special matrices. A matrix belongs to S if and only if its characteristic polynomial

$$\det(M - tI_n) = a_0 + a_1 t + ... + a_n t^n$$

has 0 as a single root. (Note that $a_0 = \det(M) = 0$.) This happens if and only if $a_1 \neq 0$. In other words, a matrix belongs to S if and only if $a_1(M) \neq 0$. The function $M \to a_1(M)$ is a polynomial. A nontrivial polynomial cannot vanish on an open subset of X. This means that a nontrivial polynomial is nonzero on a dense subset of X. We just have to show that a_1 is in fact a nontrivial polynomial on X. That is, we just have to show that some special matrix has nonzero a_1. Let $A = (n-1, -1, ..., -1)$ and consider the circulent matrix M_A. A calculation similar to the one done at the end of the last section shows that $\lambda_k = n$ for $k \neq 0$. That is, the eigenvalues of M_A are $0, n, ..., n$. Hence $a_1 = n^{n-1} \neq 0$. ♠

Lemma 0.2 *Let M be any special matrix. Then the entries of M^* are all the same. That is, all the cofactors of M have the same value.*

Proof: We are trying to prove an equality between the cofactors of M, and these cofactors vary continuously with the entries of M. So, it suffices to prove the result on a dense subset of the set of special matrices. We will show that the result holds for any nice special matrix. This suffices, by the previous result.

From the fact that $\det(M) = 0$ and Equation 9 we see MM^* is the zero matrix. This means that each column of M^* is in the kernel of M. Since M is nice, the kernel of M is precisely the span of $(1, ..., 1)$. Hence, all the entries in any column of M^* are the same. But M^* is also a symmetric matrix. Hence, all the entries in any row of M^* are the same. Hence, all entries of M^* are the same. ♠

Determinant Formula Let $m \geq n$. Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. (If $m > n$ then A is wider than it is tall, and B is taller than it is wide.) Let S denote the set of all n element subsets of $\{1, ..., m\}$. Given $s \in S$, let A_s denote the minor of A corresponding to the columns of A that belong to s. Likewise define B_s, with respect to rows. The basic formula is

$$\det(AB) = \sum_{s \in S} \det(A_s) \det(B_s). \tag{10}$$

When $m = n$ this reduces to the familiar formula $\det(AB) = \det(A)\det(B)$.

Following the suggestion in the book, consider the identity

$$\begin{bmatrix} I_m & 0_n \\ A & I_n \end{bmatrix} \begin{bmatrix} -I_m & B \\ A & 0_n \end{bmatrix} = \begin{bmatrix} -I_m & B \\ 0_n & AB \end{bmatrix} \tag{11}$$

Here I_m is the $m \times m$ identity matrix and 0_n is the $n \times n$ zero matrix. The last of these matrices has determinant $-\det(AB)$ and the first of the matrices has determinant 1. Hence,

$$\det(AB) = -\det \begin{bmatrix} -I_m & B \\ A & 0_n \end{bmatrix}. \tag{12}$$

The right hand side of Equation 12 is the sum of $(m + n)!$ terms. Each term corresponds to a choice of $(m + n)$ elements of the matrix, such that every row and column contains exactly one element. The choices which have elements in 0_n do not contribute. So, we always choose n elements from A and n elements from B, and $m-n$ elements from I_m.

There is a further constraint. Let τ be some choice of elements corresponding to a term in the determinant sum. There is some subset $s(A,\tau) \in S$ corresponding to the columns of A where the elements live. Likewise, there is some subset $s(B,\tau) \subset S$ corresponding to the rows of B where the elements live. Here is the key observation: If $s(A,\tau) \neq s(B,\tau)$ then there is no way the remaining $m-n$ elements of τ, which live in I_m, can all be nonzero. So, the nonzero terms only correspond to the cases where $s(A,\tau) = s(B,\tau)$.

Let T_s denote the set of terms τ in the determinant formula such that $s(\tau, A) = s(\tau, B) = s$. Let $[\tau]$ denote the term corresponding to τ in the determinant. That is, $[\tau]$ is the product of the elements in τ, multiplied by ± 1 according to the sign of the permutation represented by τ. We have

$$-\det(AB) = \sum_{s \in S} \sum_{\tau \in T_s} [\tau]. \tag{13}$$

The set T_s consists in exactly $(n!)^2$ terms. We can choose any two permutations of $\{1, ..., n\}$ and use these permutations to select elements of the minors A_s and B_s. Our sum of $(n!)^2$ terms factors into a product of two sums of $n!$ terms each, because each of the permutations may be freely chosen. Hence,

$$\sum_{\tau \in T_s} [\tau] = \pm \det(A_s) \det(B_s). \tag{14}$$

Working out the sign is a bit tedious, but some trial and error (based on the possible parities of m and n) is enough to convince you that the sign is always (-1). Putting everything together gives the main formula.