Here are some optional HW problems. They might be very challenging. If you like, you can do two of these and substitute them for two of the HW problems on the normal assignment.

1. The *Hilbert cube* is the space of sequences $\{a_n\}$ such that $0 \leq a_n \leq 2^{-n}$ for all $n = 0, 1, 2, ...$ The metric on this space is defined like this:

$$d(\{a_n\}, \{b_n\}) = \sum_{n=0}^{\infty} |a_n - b_n|.$$

This series always converges because it is majorized by a geometric series. Put another way, the Hilbert cube is the product $[0,1] \times [0,1/2] \times [0,1/4] \times ...$ Prove that the Hilbert cube is compact. Hint: try something vaguely like "divide-and-conquer".

2. *Hyperspace* is defined to be the set X of closed subsets of the unit square $[0,1]^2$ equipped with the following metric. Given two closed subsets $C_1, C_2 \subset [0,1]^2$, define $d(C_1, C_2)$ to be the infimum over all ϵ such that each point of C_1 is within ϵ of some point of C_2, and *vice versa*. Intuitively, $d(C_1, C_2)$ is small when a slightly fattened version of C_1 contains C_2 and *vice versa*. Prove that X is compact.

3. Say that a topological space is *totally path disconnected* if it is impossible to join two distinct points in the space with a continuous path. Construct a subset of $\boldsymbol{R}^2$ which is totally path disconnected and yet connected. Hint: Watch the movie *Being John Malkievich* and think about the topologist's sine curve during the scene when Malkievich crawls through the tube into his own brain.

4. Prove that the 2-adic solenoid is connected but not path connected. This problem requires you to define the thing clearly, for starters.