Math 153 Midterm 1 Solutions: Prof. Schwartz

1. First, $a = eae^{-1}$ for all $a \in G$. So $a \sim a$. Second, if $a \sim b$ then $a = gbg^{-1}$. But then $b = g^{-1}a(g^{-1})^{-1}$. So $b \sim a$. Third, if $a \sim b$ and $b \sim a$ then $a = gbg^{-1}$ and $b = hch^{-1}$. Combining these, we get $a = g(hch^{-1})g^{-1} = (gh)c(gh)^{-1}$. Therefore $a \sim c$. That is what you need for an equivalence relation.

2. Suppose first that $K \subset H$. We want to show that $\phi^{-1}(\overline{H}) = H$. Pick $h \in H$. Then $h \in \phi^{-1}(\phi(h))$. This shows that $h \in \phi^{-1}(\overline{H})$. Hence $H \subset \phi^{-1}(\overline{H})$. Still assuming that $K \subset H$, choose $h \in \phi^{-1}(\overline{H})$. We have $\phi(h) \in \overline{H}$. So $\phi(h) = \phi(a)$ for some $a \in H$. But then $\phi(ah^{-1}) = e$. So $ah^{-1} \in K$. But $K \subset H$. Hence $ah^{-1} \in H$. This shows that $h \in H$. So, we have shown that $\phi^{-1}(\overline{H}) \subset H$. Combining this with the knowledge that $H \subset \phi^{-1}(\overline{H})$, we conclude that $H = \phi^{-1}(\overline{H})$.

Conversely, suppose that $H = \phi^{-1}(\overline{H})$. This means, in particular, that

$$K = \phi^{-1}(e) \subset \phi^{-1}(\overline{H}) = H.$$

So $K \subset H$.

3: There are 3 things we need to show:

1. HK is closed under taking products.
2. HK is closed under taking inverses.
3. $g(HK)g^{-1} \subset HK$.

In the proof all h's belong to H and all k's belong to K. For the first item $(h_1k_1)(h_2k_2) = h_1(k_1h_2)k_2$ by the associative law. Note that

$$k_1h_2 \in k_1H = Hk_1.$$

The equality uses the fact that the left cosets of H equal the right cosets of H, and tells us that $k_1h_2 = h_3k_1$. Therefore

$$h_1(k_1h_2)k_2 = h_1(h_3k_1)k_2 = h_4k_3 \in HK.$$

This proves the first item.

For the second item,

$$(hk)^{-1} = k^{-1}h^{-1} \in k^{-1}H = Hk^{-1},$$

again using the fact that the left cosets of H equal the right cosets of H. So $(hk)^{-1} = h'k^{-1} \in HK$. This proves the second item.

For the third item, choose any $hk \in HK$, and any $g \in G$. We have

$$g(hk)g^{-1} = (ghg^{-1})(gkg^{-1}) \in HK.$$

The last containment uses the fact that both H and K are normal. This shows that $g(HK)g^{-1} \subset HK$ for all $g \in G$.

4. The groups are not isomorphic. There are various ways to see this and here is one way. All the flips in D_4 have order 2. So D_4 has more than one element of order 2. On the other hand, Q only has one element of order 2, namely -1. To see this, note that $I^2 = J^2 = K^2 = -1$, so none of these elements has order 2. Also $(-I)^2 = (-J)^2 = (-K)^2 = -1$, so none of these elements has order 2 either.

5a. There is a bijection from $\boldsymbol{Z} \to 2\boldsymbol{Z}$, namely $\phi(n) = 2n$. Let's use ϕ to give a bijection from G_1 to G_2. Given some $g_1 \in G_1$, define

$$\Phi(g_1) := g_2 = \phi \circ g_1 \circ \phi^{-1}.$$

The map ϕ^{-1} is a bijection from $2\boldsymbol{Z}$ to $\boldsymbol{Z}$, and then g_1 is a bijection from $\boldsymbol{Z}$ to itself, and then ϕ is a bijection from $\boldsymbol{Z}$ to $2\boldsymbol{Z}$. So, g_2 is a bijection from $2\boldsymbol{Z}$ to $2\boldsymbol{Z}$. Hence $g_2 \in G_2$. We need to prove that ϕ is a homomorphism, ϕ is injective, and ϕ is surjective.

For the first item, choose $a, b \in G_1$.

$$\Phi(ab) = \phi \circ (a \circ b) \circ \phi^{-1} = (\phi \circ a \circ \phi^{-1}) \circ (\phi \circ b \circ \phi^{-1}) = \Phi(a)\Phi(b).$$

This proves that Φ is a homomorphism.

For the second item, Let I_1 and I_2 respectively be the identity elements in G_1 and G_2. Suppose that $\Phi(g_1) = I_2$. Then

$$g_1 = \phi^{-1}g_2\phi = \phi^{-1}I_2\phi = \phi\phi^{-1} = I_1.$$

So, the kernel of Φ is trivial. Hence Φ is injective.

For the third item, choose $g_2 \in G_2$. Let $g_1 = \phi^{-1} \circ g_2 \circ \phi$. Then

$$\Phi(g_1) = \phi \circ (\phi^{-1} \circ g_2 \circ \phi) \circ \phi^{-1} = (\phi \circ \phi^{-1}) \circ g_2 \circ (\phi^{-1} \circ \phi) = g_2.$$

This shows that Φ is onto.

5b. Let's change notation, and let G denote the group of permutations of the integers. Let $\boldsymbol{Z}_0$ denote the even integers and let $\boldsymbol{Z}_1$ denote the odd integers. Let G_j be the group of permutations of $\boldsymbol{Z}_j$. By part a, there is an isomorphism $\Phi_0 : G_0 \to G$, The same argument gives an isomorphism $\Phi_1 : G_1 \to G$. Let $H \subset G$ be the subset of permutations h such that $h(\boldsymbol{Z}_j) = \boldsymbol{Z}_j$ for $j = 0, 1$. Clearly H is closed under products and inverses. So, H is a subgroup. Let h_j be the restriction of h to $\boldsymbol{Z}_j$. Define

$$\Phi(h) = (\Phi_0(h_0), \Phi_1(h_1)).$$

This gives a map from G into $G \times G$. The map Φ is a homomorphism because $(h \circ h')_j = h_j \circ (h')_j$ for $j = 0, 1$. We are just composing the permutations on each of $\boldsymbol{Z}_0$ and $\boldsymbol{Z}_1$.

To show that Φ is injective, suppose that $\Phi(h)$ is the identity. Then both h_0 and h_1 are the identity. But then h is the identity.

To show that Φ is surjective, pick any pair $(g_0, g_1) \in G \times G$. Let h be the permutation such that $h_j = \Phi_j^{-1}(g_j)$. In other words, use g_j to determine the action of h on $\boldsymbol{Z}_j$. By construction $\Phi(h) = (g_0, g_1)$. So, Φ is surjective.

All in all, we produced an isomorphism from H to $G \times G$.

6. Let H be a nontrivial subgroup of $\boldsymbol{Z}$. We claim that H consists of powers of a single element. That is, $H = \{nd | n \in \boldsymbol{Z}\}$ for some nonzero $d \in \boldsymbol{Z}$. Once we know this, there is an isomorphism from H to $\boldsymbol{Z}$, namely the map which sends nd to n.

To prove our claim, let d denote the smallest positive element of H. Since H is nontrivial and a subgroup, $d > 0$. If there is some $a \in H$ such that d does not divide a, then we can write $a = md + r$ where $r > 0$ and $r < d$. But then $r \in H$, and this contradicts the choice of d. So all elements of H are multiples of d. At the same time, any integer multiple of d belongs to H, because H is a subgroup. This proves our claim about H, and finishes the problem.