Approximating π: In class, we approximated π by a sequence of constructible numbers. A few of you expressed interest in some other ways of approximating π. These notes (which are not generally related to the class material) explain Leibniz's formula,

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}, \tag{1}$$

and Machin's formula

$$\pi = 16\sum_{n=0}^{\infty}\frac{(-1)^n(1/5)^{2n+1}}{2n+1} - 4\sum_{n=0}^{\infty}\frac{(-1)^n(1/239)^{2n+1}}{2n+1}. \tag{2}$$

Proof of Leibniz's Formula: Let $x \in (0,1]$. Integrate both sides of

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

to get the Taylor Series for $\tan^{-1}(x)$:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}. \tag{3}$$

Setting $x = 1$ and using $\pi = 4\tan^{-1}(1)$, we get Leibniz's formula. ♠

Proof of Machin's Formula: A direct calculation shows that

$$(5+i)^4 = (239+i)(2+2i). \tag{4}$$

When complex numbers are multiplied, their arguments add. Hence

$$4\ \arg(5+i) = \arg(239+i) + \arg(2+2i). \tag{5}$$

Note that $\arg(x+iy) = \tan^{-1}(y/x)$. Therefore, by Equation 5,

$$\pi = 4\tan^{-1}(1) = 4\ \arg(2+2i) = 16\ \arg(5+i) - 4\ \arg(239+i) =$$

$$16\tan^{-1}(\frac{1}{5}) - 4\tan^{-1}(\frac{1}{239}).$$

Combining this with Equation 3, we get Machin's formula. ♠

To study the numerical efficiency of these formulas define

$$A_N(x) = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{6}$$

This is just a truncation of the Taylor series in Equation 3.

Leibniz's Formula in Action: For notational convenience, let $A_k = A_k(1)$. We have $A_k < \pi < A_{k+1}$ when k is odd. A calculation shows

$$3.14059... = A_{999} < \pi < A_{1000} = 3.14259...$$

So, after adding up about 500 fractions we get $\pi = 3.14...$ but only incomplete information about the 4th digit. Leibniz's formula has a special beauty to it, but it is painfully slow.

Machin's Formula in Action: Define

$$B_n = 16A_n(1/5) - 4A_n(1/239).$$

We have $B_k < \pi < B_{k+1}$ when k is odd. The short calculation

$$3.141592652... = B_5 < \pi < B_6 = 3.141592653...$$

already pins down the first 9 digits of π. The estimate $B_{71} < \pi < B_{72}$ pins down the first 100 digits. Machin's formula keeps up its blistering pace in the long run. Consider the ratio

$$R(k) = \frac{B_{k-1} - \pi}{\pi - B_k}$$

Here are some sample values

$$R(1) = 42.8541... \quad R(100) = 25.2488... \quad R(10000) = 25.00249988....$$

In fact B_{k+1} is always more than 25 times closer to π than is B_k.

Conclusion: If you ever meet a creature from outer space and have just one chance to make a good first impression on behalf of the human race, show it Machin's formula.