M1720 HW1

1: Let X be the space of equivalence classes of $[0,1]^2$ with respect to the relations $[x,0] \sim [x,1]$ and $[0,y] \sim [1,y]$ and $[0,0] \sim [0,1] \sim [1,1] \sim [1,0]$. In other words, we get X by gluing opposite sides of $[0,1]^2$. There are two ways to put a topology on X. The first way is to use the quotient topology.

Here is the second way. Say that a *path* between points $A, B \in X$ a union $L = \gamma_1 \cup ... \cup \gamma_n$ of directed line segments in $[0,1]^2$ such that

- A is the initial endpoint of γ_1 and B is the final endpoint of γ_n.
- The final endpoint of γ_i is in the same equivalence class as the initial endpoint of γ_{i+1}, for each $i = 1, ..., (n-1)$.

Define the distance between A and B to be the infimum of all the lengths of paths between A and B. Prove that the second method makes X into a metric space, and prove that the two methods give homeomorphic spaces.

Hint: The two spaces have the same underlying set. Figure out what the metric balls are and show that they also serve as a basis for the quotient topology. Use this fact to show that the identity map between the two spaces is a homeomorphism.

2: Prove that the surface of a cube is homeomorphic to the surface of a regular tetrahedron.

3: Let V be the set of all C^2 functions of $\boldsymbol{R}^2$ whose mixed partials commute at $(0,0)$. That is $\partial_x \partial_y F(0,0) = \partial_y \partial_x F(0,0)$. Prove that V is a vector space that contains all functions of the form $F(x,y) = f(x)g(y)$, where f and g are C^2.

4: Let $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function such that $F(x,0) = F(0,y) = 0$ for all x, y and such that $\partial_x \partial_y F(0,0) = 0$. Prove that

$$\lim_{t \to 0} \frac{F(t,t)}{t^2} = 0.$$

Hint: The mean value theorem is probably helpful here.

5: Let $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function such that $F(x, 0) = F(0, y) = 0$ for all x, y and such that $\partial_y \partial_x F(0, 0) = C$. Prove that

$$\lim_{t \to 0} \frac{F(t, t)}{t^2} = C.$$

Hint: Apply Problem 4 to the new function $G(x, y) = F(y, x) - Cxy$. Now put everything together and prove that the mixed partials of a C^2 function commute.