M1720 HW2

1: Recall that a smooth manifold is a topological manifold together with a maximal atlas $\mathcal{A}$ of coordinate charts such that all the overlap functions relative to these are diffeomorphisms. We say that the coordinate charts are *compatible*.

Suppose that X is a topological manifold and it has some covering $\mathcal{C}$ by compatible coordinate charts. Here $\mathcal{C}$ need not be maximal. Suppose also that X has two other coordinate charts ϕ_1 and ϕ_2 which are compatible with all the charts in $\mathcal{C}$. Prove that ϕ_1 and ϕ_2 are compatible with each other. (Hint: checking that a homeomorphism is a diffeomorphism is a local thing.)

The point of this exercise is that you can automatically extend an atlas of charts with smooth overlap functions to a maximal atlas. This is the labor-saving device we are using repeatedly in class.

2: A *triangle* in a smooth 2-manifold M is a subset $T \subset M$ such that T is diffeomorphic to a triangle in $\boldsymbol{R}^2$. This means that T is contained in some open set U and there is a diffeomorphism $\phi : U \to \boldsymbol{R}^2$ such that $\phi(T)$ is a triangle in the ordinary sense. A *triangulation* of M is a way to write M as a union of finitely many triangles such that two triangles are either disjoint, or intersect in a common vertex, or intersect in a common edge. Prove that the torus has a triangulation made from 7 vertices.

3: a: Prove that the projective plane has a triangulation with 6 vertices. (The bottom of the next page has a hint, which I tried to obscure so you have to really look to see it.)
b. Prove that the projective plane minus a single point is homeomorphic to an open Moebius band. For a model of an open Moebius band, take $[0,1] \times (0,1)$ mod the relation that $(0,y) \sim (1, 1-y)$.

4: Let $G(4,2)$ denote the set of 2-planes through the origin in $\boldsymbol{R}^4$. Define the distance between two such 2-planes Π_1 and Π_2 as follows. Let C_k denote the intersection of Π_k with the unit sphere centered at the origin in $\boldsymbol{R}^4$. Then the distance from Π_1 to Π_2 is the infimal ϵ such that every point of C_1 is within ϵ of C_2 and *vice versa*. Prove that this makes $G(4,2)$ into a metric space. Bonus challenge: Prove that $G(4,2)$, with this metric, is a topological manifold. $G(4,2)$ is an example of a *Grassmannian manifold*.

5: The *Veronese embedding* gives a map from $\boldsymbol{RP}^2$ to $\boldsymbol{RP}^5$. In homogeneous coordinates, this map is given by

$$V : [x : y : z] \to [x^2 : y^2 : z^2 : xy : yz : zx].$$

Prove that this map is well-defined, injective, and a homeomorphism onto its image in $\boldsymbol{RP}^5$. Prove also that the image is disjoint from some hyperplane, so that really we can think of $V(\boldsymbol{RP}^2)$ as a subset of $\boldsymbol{R}^5$. Explain briefly how this embedding might generalize to a map from $\boldsymbol{RP}^n$ to $\boldsymbol{RP}^N$, where $N = \binom{n+1}{2} - 1$.

The thing I've never liked about hints given on the bottom of pages is that your eyes might accidently platonic stray over it before you can stop yourself and then solids you see the hint without wanting to. This approach has tried to hide the hint in the middle of the text. You have to really read it to see the important two words.