Metric Space Worksheet

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This is a worksheet designed to bridge the gap between Math 1010 and Math 1130 at Brown University. Just read it and do all 53 execises.

1 Basic Definition

A *metric space* is a set X together with a function $d : X \times X \to \boldsymbol{R}$ satisfying the following properties.

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$. (Symmetry)
3. $d(x, y) + d(y, z) \geq d(x, z)$. (Triangle Inequality)

These axioms are meant to hold for all points $x, y, z \in X$. Often we will denote a metric space as a pair (X, d) where X is the set and d is the metric on X. Sometimes we will forget to mention d explicitly.

Examples: Here are some examples which indicate some of the variety of metric spaces.

- X is a single point $\{x\}$, and $d(x, x) = 0$.
- X can be any set. Define $d(x, x) = 0$, and $d(x, y) = 1$ when $x \neq y$. This is often called the *discrete metric*.
- $X = \boldsymbol{R}$, the set of real numbers, and the metric is $d(x, y) = |x - y|$. This example is the main topic of Math 1010. All example involving $\boldsymbol{R}$ below implicitly use this metric.

- $X = \boldsymbol{R}^n$, and the metric is

$$d(X, Y) = \sqrt{(X - Y) \cdot (X - Y)}.$$

Here $(\cdot)$ denotes the dot product: If $V = (v_1, ..., v_n)$ and $W = (w_1, ..., w_n)$ then $V \cdot W = \sum_{i=1}^{n} v_i w_i$. If you think about it, the metric here is what you would get from the multi-dimensional Pythagorean Theorem. This is called the *Euclidean metric*, and also sometimes called the ℓ_2-*metric*.

- $X = \boldsymbol{R}^n$ and $d(X, Y) = \max_i |x_i - x_i|$. Here $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$. This is often called the ℓ_∞-*metric*.

- $X = \boldsymbol{R}^n$ and (with the same notation as in the previous example) $d(X, Y) = \sum_{i=1}^{n} |x_i - x_i|$. This is often called the ℓ_1-*metric*.

- X is the set of infinite sequences $\{a_n\}$ of real numbers, subject to the condition that

$$\sum_{n=1}^{\infty} a_n^2 < \infty.$$

The metric is given by

$$d(\{a_n\}, \{b_n\}) = \Big(\sum_{n=1}^{\infty} (a_n - b_n)^2\Big)^{1/2}.$$

This space is known as ℓ_2.

- $X = \boldsymbol{Z}$ with the metric $d(m, n) = 2^{-n}$ where n is the maximum number of powers of 2 dividing $m - n$. For instance $d(3, 51) = 2^{-4}$ because $16 = 2^4$ divides $48 = 51 - 3$ but $32 = 2^5$ does not. If $m = n$ we set $d(m, n) = 0$. This metric is called the 2-*adic metric*.

- $X = \boldsymbol{Z}$ with the same metric as in the previous case, except that you use a prime p in place of 2. This is called the p-*adic metric* on $\boldsymbol{Z}$.

- X is the vertex set of a connected graph (i.e. a collection of vertices connected by edges) and $d(v, w)$ is the minimum number of edges in a path in the graph that joins v to w.

Exercise 1: Prove, in each of the above cases, that the object really is a metric space.

2 Open Sets

Let (X, d) be a metric space. Given $p \in X$ and $\epsilon > 0$, we define

$$B_\epsilon(p) = \{q |\ d(p, q) < \epsilon\}. \tag{1}$$

The set $B_\epsilon(p)$ is called the *open ball of radius* ϵ *about* p. More abstractly, an *open ball* is defined to be $B_\epsilon(p)$ for some choice of p and some $\epsilon > 0$. The reason for the terminology comes from the case of $\boldsymbol{R}^n$ with the Euclidean metric.

Exercise 2: Suppose that B_1 and B_2 are open balls and $p \in B_1 \cap B_2$ is a point in the intersection. Prove that there is a third open ball B_3 such that $p \in B_3$ and $B_3 \subset B_1 \cap B_2$. The solution to this uses the triangle inequality in a crucial way.

A subset $U \subset X$ is called *open* if it has following property. If $p \in U$ then there is some open ball B such that $p \in B$ and $B \subset U$. The empty set (which technically does satisfy this property) is declared to be open.

Exercise 3: Prove that any union (even an infinite union) of open subsets of X is again open in X.

Exercise 4: Prove that any finite intersection of open subsets of X is an open subset of X. Note that you can just prove this for an intersection of 2 open sets and then use induction for the general case. Exercise 2 should come in handy here.

Exercise 5: Show by way of example that you can have an infinite intersection of open subsets of a metric space which is not open.

Exercise 6: Prove that an open ball is open according to the definition given.

Exercise 7: Suppose $S \subset X$. We can think of S as a metric space in its own right by restricting the metric d to S. Let U_S be an open subset of S. Prove that there is an open subset U of X such that $U_S = U \cap S$. Hint: Each point $p \in U_S$ is contained in an open ball $B'_\epsilon(p) \subset S$. But $B'_\epsilon(p) = B_\epsilon(p) \cap S$, where $B_\epsilon(p)$ is the corresponding open ball in X. Let U be the union of these X-balls.

3 Continuous Maps

Let (X_1, d_1) and (X_2, d_2) be metric spaces. By a *map* $f : X_1 \to X_2$ we mean a rule whose input is $x_1 \in X_1$ and whose output is $x_2 = f(x_1) \in X_2$. The map f is meant to be defined on all points of X_1.

Given any subset $S_2 \subset X_2$ we define $f^{-1}(S_2)$ to be the subset $S_1 \subset X_1$ such that $x_1 \in S_1$ if and only if $f(x_1) \in S_2$. In other words, $f(S_1) \subset S_2$ and S_1 is the largest subset of X_1 with this property.

The map f is said to be *continuous* if the following implication always holds: If U_2 is an open subset of X_2 then $U_1 = f^{-1}(U_2)$ is an open subset of X_1. This definition looks rather different from the usual definition of continuity given in Math 1010. Let's reconcile this.

Exercise 8: Say that the map f is *old-school continuous* if it has the following property: For all $x_1 \in X_1$ and all $\epsilon > 0$ there is a $\delta > 0$ such that if $d_1(x_1, x_1') < \delta$ then $d_2(x_2, x_2') < \epsilon$. Here $x_2 = f(x_1)$ and $x_2' = f(x_1')$. Prove that f is old-school continuous if and only if f is continuous.

Exercise 9: In case $X_1 = X_2 = \boldsymbol{R}$ and $d_1(x, y) = d_2(x, y) = |x - y|$ show that the old-school definition of continuity coincides with your memory of how continuous functons were defined in Math 1010.

Now suppose that X_3 is a third metric space. If we have two given maps $f : X_1 \to X_2$ and $g : X_2 \to X_3$ the composition $g \circ f$ is a map from X_1 to X_3. It is defined like this: $g \circ f(x) = g(f(x))$.

Exercise 10: Prove that the composition of continuous maps is continuous. You will find that this is easier using the new definition of continuity.

Here are three more definitions about a map $f : X_1 \to X_2$.

- f is *injective* if $f(x) = f(y)$ only when $x = y$. This is a.k.a. *one-to-one*.
- f is *surjective* if $f(X_1) = X_2$. This is a.k.a. *onto*.
- f is a *bijection* if f is both injective and surjecrive.

$f : X_1 \to X_2$ is a *homeomorphism* if f is a bijection and both maps f and f^{-1} are continuous. Here f^{-1} is defined so that $f \circ f^{-1}$ is the identity map. Two metric spaces X_1 and X_2 are *homeomorphic* if there is a homeomorphism between them.

Exercise 11: Let $X_1 = \boldsymbol{R}^2$ with the Euclidean metric. Let $X_2 \subset \boldsymbol{R}^2$ be any open ball with respect to the Euclidean metric. Prove that X_1 and X_2 are homeomorphic. You will need to construct a continuous map between them.

Exercise 12: Let X_1 be $\boldsymbol{R}^2$ equipped with the Euclidean metric. Let X_2 be $\boldsymbol{R}^2$ equipped with the ℓ_∞ metric. Prove that X_1 and X_2 are homeomorphic.

Exercise 13: This one is pretty hard if you are a newcomer to metric spaces. Let $X = \boldsymbol{R}^2$ equipped with the Euclidean metric. Let $p_1, ..., p_n$ be any n distinct points in X and let $q_1, ..., q_n$ be any other n distinct points. Prove that there is a homeomorphism $h : X \to X$ such that $h(p_k) = q_k$ for $k = 1, ..., n$. Hint: try to do this one at a time: Connect p_1 to q_1 by a "tunnel" which avoids the other points and try to modify things inside the tunnel. Then move on to the next pair...

4 Some Point-Set Topology

Let (X, d) be a metric space. A subset $C \subset X$ is *closed* if and only if the complement $X - C$ is open.

Exercise 14: Prove that the arbitrary intersection of closed sets is closed and the finite union of closed sets is closed.

Given any subset $S \subset X$, the *closure* of S is the intersection

$$\overline{S} = \bigcap_{C \subset S} C \tag{2}$$

taken over all closed sets which contain S. By Exercise 13, $\overline{S}$ is a closed set.

Exercise 15: Prove that $\overline{S}$ is the smallest closed set containing S. In other words, if C is another closed set such that $S \subset C$, then $\overline{S} \subset C$ as well. This result pretty much follows straight from the definition.

Exercise 16: Prove that S is closed if and only if $\overline{S} = S$.

A subset $S \subset X$ is *dense* in X if $\overline{S} = X$.

Exercise 17 Let $\boldsymbol{Q}$ be the set of rationals, contained inside the metric space $\boldsymbol{R}$ of reals. Prove that $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$.

A *sequence* in X is a list of points $p_1, p_2, p_3, ...$. Such a sequence is often written as $\{p_k\}$. The sequence could either be finite or infinite, but we're going to talk about infinite sequences. A point $x \in X$ is an *accumulation point* of the sequence $\{p_k\}$ if every open subset U which contains p contains infinitely many terms of the sequence. That is, there are infinitely many indices k such that $p_k \in U$.

Exercise 18: Here is an alternative definition of accumulation points: x is an *old-school accumulation* point of $\{p_k\}$ if, for every $\epsilon > 0$ there are infinite many indices k such that $d(x, p_k) < \epsilon$. Prove that x is an old-school accumulation point for $\{p_k\}$ if and only if x is an accumulation point for $\{p_k\}$.

Exercise 19: Suppose $\{p_k\} \in S$, which means that all the points in the sequence lie in S. Prove that all the accumulation points of $\{p_k\}$ lie in $\overline{S}$.

Exercise 20: Give an example of a sequence in $\boldsymbol{R}$ which (with respect to the usual metric) has every point of $\boldsymbol{R}$ as an accumulation point.

Exercise 21: Prove that the set of accumulation points of a sequence in X is a closed subset of X.

A sequence $\{p_k\}$ *converges* to $x \in X$ if the set of accumulation points of $\{p_k\}$ is just the single element $\{x\}$. Such a sequence is called *convergent*.

Exercise 22: Suppose $f : X_1 \to X_2$ is a continuous map and $\{p_k\}$ is a sequence that converges to $x_1 \in X_1$. Prove that the sequence $\{f(p_k)\}$ converges to $f(x_1)$ in X_2. Hence continuous maps carry convergent sequences to convergent sequences.

Let (X, d) be a metric space. A sequence $\{p_k\}$ is called *Cauchy* if the following is true. For every $\epsilon > 0$ there is some N such that if $m, n > N$ then

$d(p_m, p_n) < \epsilon$.

Exercise 23: Prove that a convergent sequence in a metric space is Cauchy.

Exercise 24: Give an example of a metric space which has a Cauchy sequence that is not convergent. Hint: Think about metric spaces you already know well.

5 Metric Completions

We saw in Exercise 24 that a metric space can have a Cauchy sequence which is not a convergent sequence. A metric space is called *complete* if every Cauchy sequence is convergent.

Exercise 25: The real numbers are often constructed in such a way that they have the *least upper bound property*: Every bounded set S of real numbers is such that there is a number called $\alpha = \sup S$ with two properties:

- $r \leq \alpha$ for all $r \in S$.
- If $\alpha' < \alpha$ then there is some $s \in S$ such that $\alpha' < s$.

Use the least upper bound property to prove that $\boldsymbol{R}$ is complete. Hint: It might also be useful to know that $\boldsymbol{R}$ also has the *greatest lower bound property*, which is defined similarly to the least upper bound property and which follows from it and symmetry.

Let (X, d) and (X^*, d^*) be metric spaces. A map $f : X \to X^*$ is called an *isometric injection* if $d^*(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. This definition implies that f is an injective map. The metric space (X^*, d^*) is called a *metric completion* of (X, d) if there is an isometric injection $f : X \to X^*$ such that $f(X)$ is dense in X^*. Usually we identify X with $f(X)$ and just think of it as a subset of X^*.

Exercise 26: Suppose that X is already complete and $f : X \to X^*$ is a metric completion. Prove that f is both a bijection and an isometry. In other words, essentially $X = X^*$ in this case, up to how we name the points.

Exercise 27: Prove that $\boldsymbol{R}$ is a metric completion of $\boldsymbol{Q}$, using the inclusion map $\iota : \boldsymbol{Q} \to \boldsymbol{R}$.

An Extended Example: Here is a construction of what is called the 2-*adic completion* of $\boldsymbol{Z}$. Recall that $\boldsymbol{Z}/n$ is the group of integers mod n. Recall that the powers of 2 are $1, 2, 4, 8, 16, ...$ Consider the infinite chain of maps

$$\cdots \stackrel{\pi}{\to} \boldsymbol{Z}/16 \stackrel{\pi}{\to} \boldsymbol{Z}/8 \stackrel{\pi}{\to} \boldsymbol{Z}/4 \stackrel{\pi}{\to} \boldsymbol{Z}/2.$$

In all cases, π is the "reduction" map. For instance, if $5 \in \boldsymbol{Z}/8$ then we have$\pi(5) = 1 \in \boldsymbol{Z}/4$. A 2-*adic integer* is a sequence $\{b_n\}$ where $b_n \in \boldsymbol{Z}/2^n$ and $\pi(b_n) = b_{n-1}$ for all n. Let $\boldsymbol{Z}_2$ denote the set of 2-adic integers. There is a natural metric on $\boldsymbol{Z}_2$. For two distinct 2-adic integers $\{a_n\}$ and $\{b_n\}$, we define $d(\{a_n\}, \{b_n\}) = 2^{-n}$, where n is the largest integer such that $a_n = b_n$. If the two sequences are equal we define their distance to be 0, as always.

Exercise 28: Prove that $(\boldsymbol{Z}_2, d_2)$ is complete.

Exercise 29: There is a natural map $f : \boldsymbol{Z} \to \boldsymbol{Z}_2$. $f(k) = \{b_n\}$ where b_n is the representative of k in $\boldsymbol{Z}/2^n$. Prove that f is an isometric injection from $(\boldsymbol{Z}, d)$ to $(\boldsymbol{Z}_2, d_2)$, where d is the 2-adic metric discussed in §1.

Exercise 30: Continuing from the previous exercise, prove that $f(\boldsymbol{Z})$ is dense in $\boldsymbol{Z}_2$. This combines with the Exercises 28 and 29 to show that $\boldsymbol{Z}_2$ is a metric completion of $\boldsymbol{Z}$ with respect to the 2-adic metric on $\boldsymbol{Z}$.

Exercise 31: Generalize the construction above to the case of an arbitrary prime p, and behold the p-adic completion of $\boldsymbol{Z}$.

A General Construction: Here is a general construction of a metric completion for any metric space X. Let X' denote the set of Cauchy sequences of X. Given two members $\{a_n\}$ and $\{b_n\}$ of X' write $\{a_n\} \sim \{b_n\}$ if the shuffled sequence $a_1, b_1, a_2, b_2, a_3, b_3, ...$ is also Cauchy.

Exercise 32: Prove that $\sim$ is an equivalence relation on X'. This means that

- $\{a_n\} \sim \{a_n\}$.
- $\{a_n\} \sim \{b_n\}$ if and only if $\{b_n\} \sim \{a_n\}$.

- If $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$ then $\{a_n\} \sim \{c_n\}$.

Exercise 32 implies that X' is partitioned into *equivalence classes*. An equivalence class is the set of all Cauchy sequences equivalent to a given one. Let X^* be the set of equivalence classes in X'. A point of X^* has the form $[\{b_n\}]$, and this means the set of all Cauchy sequences equivalent to $\{b_n\}$.

Exercise 33: Given points $[\{a_n\}], [\{b_n\}] \in X^*$ define their distance to be

$$d^*([\{a_n\}], [\{b_n\}]) = \lim_{n\to\infty} d(a_n, b_n).$$

Prove that this definition is independent of which points in X' represent the equivalence classes. Hence d^* is well defined.

Exercise 34: Prove that d^* is a metric on X^* and that (X^*, d^*) is complete.

Exercise 35: There is a natural map $f : X \to X^*$. $f(x)$ is defined to be the equivalence class of the constant sequence $x, x, x, ...$ Prove that f is an isometric injection.

Exercise 36: Prove that $f(X)$ is dense in X^*. Combining this exercise with Exercises 34 and 35, we see that X^* is a metric completion of X.

6 Compactness

Let (X, d) be a metric space. An *open cover* of X is a collection $\{U_\alpha\}$ of open subsets of X whose union is all of X. The open cover might or might not be finite. A *subcover* of an open cover is some sub-collection of the open sets in the cover which is, itself, an open cover. The subcover is called a *finite subcover* if it has only finitely many elements.

Exercise 37: Make the interval $(0, 1)$ into a metric space using the metric $d(x, y) = |x - y|$. Prove that $(0, 1)$ has an infinite open cover with no finite subcover.

Exercise 38: A metric space X is called *unbounded* if, for every N there are points $p, q \in X$ such that $d(p, q) > N$. Another formulation is that X has a point p_0 such that for all N there is some point q such that $d(p_0, q) > N$. The two formulations are equivalent. Prove that if X is unbounded then X has an infinite cover with no finite subcover.

A metric space X is called *compact* if every cover of X has a finite subcover. Exercise 37 implies that $(0, 1)$ is not compact. Exercise 38 implies that unbounded metric spaces are not compact.

If $S \subset X$ is a subset, we can think of (S, d) as a metric space in its own right and we can then speak about whether or not S is compact. Thus, the notion of compactness applies not just to metric spaces but also to subsets of metric spaces.

Let us say a bit more about this. What are the open subset of S, considered as a metric space in its own right? They all have the form $U \cap S$ where U is open in X. See Exercise 7. Thus, if we have an open cover of S we technically have a collection of open subsets of X whose union *contains* S, but we can just forget about all the points of these open sets which do not belong to S and work entirely within S. The two perspectives are completely equivalent, though psychologically one or the other might be more useful in a given situation.

Exercise 39: Suppose $K \subset X$ is compact and $S \subset K$. Suppose also that S is a closed subset of X. Prove that S is also compact. Hint: If $\{U_\alpha\}$ is a cover of S you can add the open set $X - S$ to get a cover of K...

Theorem 6.1 $[0, 1]$ *is compact.*

Proof: Given any interval $I \subset [0, 1]$, let I_0, I_1 be the two smaller intervals obtained by cutting I in half. Starting with $I = [0, 1]$ we create a tree of intervals by recursively cutting in half. Thus I is the parent of I_0 and I_1, and I_0 is the parent of I_{00} and I_{01}. Etc. In general, each interval is the *parent* of the two intervals obtained by cutting it in half, and these smaller intervals are the *children*.

Let $\mathcal{C}$ be an open cover of I. We want to find a finite sub-cover. We will suppose this does not exist and derive a contradiction. Call a closed interval $J \subset I$ *bad* no finite union of open sets of $\mathcal{C}$ covers J. Any bad interval has a bad child. Otherwise we could just assemble the two finite covers of the children to get a finite cover of the parent.

If I is bad then we can keep cutting in half and produce an infinite sequence $J_0(= I) \supset J_1 \supset J_2, \ldots$ of nested bad intervals. The length of J_n is 2^{-n}. Hence the sequence of centers of these intervals is Cauchy. By Exercise 25, it is convergent and hence has an accumulation point p. Since $\overline{J}_n = J_n$ and the "tail end" of our sequence of centers lies in J_n we have $p \in J_n$. This is true for any n.

We have just shown that $\bigcap J_k$ is non-empty: It contains p. But p lies in some open subset U of $\mathcal{C}$. But then $J_n \subset U$ once n is sufficiently large. This makes $\{U\}$ itself a finite subcover of J_n. This is a contradiction. ♠

Exercise 40: Imitate the proof above to show that the cube $[0, 1]^n$, with respect to the Euclidean metric on $\boldsymbol{R}^n$, is compact. Prove the same thing for the huge cube $[-N, N]^n$.

Exercise 41: Prove that a compact subset of a metric space is closed. Hint: If S is not closed then there is a point $p \in \overline{S} - S$, by Exercise 16. Construct an open cover of S by taking the complements of small balls centered at p. Prove that this open cover has no finite subcover.

Heine-Borel Theorem: Combining Exercises 39 and 40, we (or, rather, you) have proved that every closed and bounded subset of $\boldsymbol{R}^n$ is compact. We also know by Exercise 38 that a compact subset of $\boldsymbol{R}^n$ must be bounded. Finally, Exercise 41 shows that a compact subset of $\boldsymbol{R}^n$ must be closed. In short a subset of $\boldsymbol{R}^n$, with respect to the Euclidean metric, is compact if and only if it is closed and bounded. This is the famous Heine Borel Theorem.

Exercise 42: Suppose $f : X_1 \to X_2$ is continuous and X_1 is compact. Prove that $f(X_1)$ is also compact. Conclude that a continuous real-valued function on a compact space is bounded. In case X_1 is a closed interval, this is a classic result, probably proved in Math 1010, that every continuous function on a closed interval is bounded.

Exercise 43: Prove that a bijection $f : X_1 \to X_2$ between metric spaces is a homeomorphism provided that it is continuous. This is a great labor-saving result when you are trying to show that a map is a homeomorphism.

7 Sequential Compactness

Now we explore an alternate definition of compactness. A metric space is called *sequentially compact* if every sequence has a convergent subsequence. This section guides you through a proof that a metric space is compact if and only if it is sequentially compact.

Exercise 44: Prove that if X is compact then X is sequentially compact. Hint: If this is false then every point is contained in an open ball which only contains finitely many terms of the sequence.

The δ in the next lemma called a *Lebesgue number* for the cover.

Exercise 45: Let $\{U_\alpha\}$ be an open cover of a sequentially compact space. Prove that there is some $\delta > 0$ such that every ball of radius δ is contained in some U_α. Such a δ is called a *Lebesgue number* for the cover. Hint: If this is false then we can find a ball B_n of radius $1/n$ that is not contained in any U_α, provided that we take these values of n to be sufficiently large. Now look at the sequence of centers of these balls.

A metric space is called *totally bounded* if, for every $\delta > 0$, there is a finite cover of X consisting of δ-balls.

Lemma 7.1 *If X is sequentially compact then X is totally bounded.*

Proof: We suppose this is false and derive a contradiction. Suppose, by induction, we have found n points $x_1, ..., x_n$ such that the distance between any two points is at least δ. Since the balls of radius δ centered at these points do not cover X we can find a new point x_{n+1} at least δ from $x_1, ..., x_n$. By induction, we can find an infinite sequence $\{x_n\}$ such that every pair of points is at least δ apart. But then this sequence has no convergent subsequence. This is a contradiction. ♠

Exercise 46: Combine Lemma 7.1 with Exercise 45 to prove that a sequentially compact metric space is compact. Combining this with Exercise 44, we see that a metric space is compact if and only if it is sequentially compact.

8 Connectedness

A metric space X is called *disconnected* if we can find two open subsets $U_1, U_2 \subset X$ such that

- U_1 and U_2 are both nonempty.
- $U_1 \cap U_2 = \emptyset$.
- $X = U_1 \cup C_2$.

We call X *connected* if it is not disconnected.

We can also speak of a subset $S \subset X$ as being connected or disconnected. The metric d on X gives rise to the same metric on S, so S is a metric space in its own right and we can ask whether or not it is connected. See the discussion above about compactness of subsets for more explanation about this.

Exercise 47: Suppose $f : X_1 \to X_2$ is a continuous map between metric spaces and X_1 is connected. Prove that X_2 is also connected.

Theorem 8.1 $[0,1]$ *is connected.*

Proof: We suppose this is false and derive a contradiction. Suppose that $[0,1] \subset U \cap V$ and $U \cap V = \emptyset$ and both U, V are non-empty. Without loss of generality, suppose $1 \in V$. Since U is both bounded and non-empty, it has a least upper bound $\alpha \in [0,1]$.

Suppose $\alpha \in U$. Then, since U is open, $\alpha + \epsilon \in U$ for some $\epsilon > 0$. But this contradicts the fact that α is the least upper bound for U.

Suppose $\alpha \in V$. Then $\alpha - \epsilon \in V$ for all $\epsilon > 0$ sufficiently small. But then such points are also upper bounds for U because no point of U also lies in V. This contradicts the fact that α is the *least* upper bound for U.

We have shown that α lies neither in U nor V, which is a contradiction. ♠

The fact that $[0,1]$ is connected is really the starting point of topology. It implies the Intermediate Value Theorem from Math 1010, for instance.

Exercise 48: Deduce the Intermediate Value Theorem from the material above. That is, suppose $f : [0,1] \to \boldsymbol{R}$ is a continuous function, then $f([0,1])$

contains the interval bounded by $f(0)$ and $f(1)$. That is, f attains all values between $f(0)$ and $f(1)$.

Exercise 49: Suppose X is a metric space and $S, T \subset X$ are connected subsets which intersect. Prove that $S \cup T$ is connected.

It turns out that there are some pretty wierd connected spaces – see Exercise 50. Here is a related definition which probably coincides better with our intuitive understanding of what "connected" means. A *path* in a metric space X is the image of a map $f : [0, 1] \to X$. The metric space X is called *path connected* if for every two points $x, y \in X$ there is a path $f : [0, 1] \to X$ such that $x = f(0)$ and $y = f(1)$.

Exercise 50: Prove that a path connected metric space is connected.

Exercise 51: Prove that open and closed cubes and balls in $\boldsymbol{R}^n$, with respect to the Euclidean metric, are path connected. Hence, by Exercise 50, they are also connected.

Here is a concept at the opposite extreme of connectedness. A metric space X is *totally disconnected* if every pair of distinct points of X can be placed inside disjoint open sets whose union covers X. A totally disconnected space is disconnected in a very strong way.

Exercise 52: Prove that $\boldsymbol{Z}_2$, the 2-adic integers, is totally disconnected.

The final exercise explores the difference between connectedness and path connectedness. If you want a real challenge, do it without the hint.

Exercise 53: Give an example of a connected metric space which is not path-connected. Big Hint: Look up the *Topologist's sine curve*. If you want something even more challenging, prove that there is a connected metric space which is totally path disconnected: No two distinct points can be connected by a path.