M1720: Primer on Topological Spaces

These notes give some basic information about topological spaces, and they end with the definition of a topological manifold.

Main Definition: A *topological space* is a set X together with a collection $\mathcal{C}$ of subsets of X which have the following properties.

- The emptyset belongs to $\mathcal{C}$.
- X itself belongs to $\mathcal{C}$.
- Arbitrary unions of members of $\mathcal{C}$ belong to $\mathcal{C}$.
- Finite intersections of members of $\mathcal{C}$ belong to $\mathcal{C}$

The sets in $\mathcal{C}$ are called *open*, and $\mathcal{C}$ is called a *topology* on X.

Sometimes it is convenient to specify a topology by specifying the *closed* sets. These are the sets whose complements are open. Arbitrary intersections of closed sets are closed and finite unions of closed sets are closed.

Bases: Suppose X is a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X with the following properties:

- Every point of X is contained in some member of $\mathcal{B}$.
- If A and B are two members of $\mathcal{B}$ and $p \in A \cap B$, then there is a member C of $\mathcal{B}$ such that $p \in C \subset A \cap B$.

The basis $\mathcal{B}$ is not necessarily a topology on X. However we define $\mathcal{C}$ to be the set of arbitrary unions of subsets of $\mathcal{B}$. You can check that the properties of $\mathcal{B}$ imply that $\mathcal{C}$ is a topology. The main thing you need to check is the finite intersection property.

Metric Spaces: Let X be a metric space. The set $\mathcal{B}$ of open balls of X forms a basis for a topology on X. In this way a metric space is naturally a topological space.

Subspace Topology: Suppose X is a topological space and Y is a subset of X. Then Y inherits a topology from X. A subset of Y is declared to be open if and only if it has the form $U \cap Y$, where U is open in X.

This definition plays well with metric spaces. If X is a metric space then Y inherits the metric from X. Now there are two ways to get a topology on Y. We can either take the induced metric on Y and then take the topology coming from the metric on Y or we can take the subspace topology on Y. Both give the same topology on Y.

Product Topology: Suppose that X and Y are topological spaces. Then the product $X \times Y$ has a natural topology. A basis for the topology on $X \times Y$ is given by sets of the form $U \times V$ where U is open in X and V is open in Y. Let's check the main property to make sure it works. Suppose that $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. Then $x \in U_3 := U_1 \cap U_2$ and $y \in V_3 := V_1 \cap V_2$. But then $U_3 \times V_3$ is a basis element that that contains (x, y) and also lies in $U_j \times V_j$ for $j = 1, 2$.

Continuous Maps: Let X and Y be topological spaces. A map $f : X \to Y$ is *continuous* if it has the following property: If V is open in Y then $f^{-1}(V)$, meaning the set-theoretic inverse, is open in X. You can check that this definition subsumes the old-school metric space definition of continuity when X and Y are metric spaces.

The composition of continuous maps is continuous. That is, if $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is continuous. This follows almost immediately from the definition.

Homeomorphisms: The map f is a homeomorphism if f is a bijection and both f and f^{-1} are continuous. In other words, f just *renames* the open sets. The set of self-homeomorphisms of a topological space forms a group under composition.

The Quotient Topology: Suppose X is a topolgical space and $\sim$ is an equivalence relation on X. Let $\overline{X} = X/\sim$ denote the set of equivalence classes. We have the quotient map $f : X \to \overline{X}$. Here $f(x)$ is just the equivalence class of x. We define a topology on $\overline{X}$ as follows. A subset $\overline{U} \subset \overline{X}$ is open if and only if the set-theoretic inverse $f^{-1}(\overline{U})$ is open in X. You can check that this defines a topology on $\overline{X}$. It is called the quotient topology.

Hausdorff Spaces: A topological space X is called *Hausdorff* if it has the following property. For any two points $p \neq q \in X$ there are open sets U, V such that $p \in U$ and $q \in V$ and $U \cap V = \emptyset$. This notion imitates some properties of metric spaces. In the metric space case, we can place any two distinct points inside disjoint open balls.

If X is a finite Hausdorff topological space, then (you can check that) every subset of X is open. So, the only interesting finite topological spaces are non-Hausdorff.

Non-Hausdorff Examples: Let me give some examples of non-Hausdorff spaces.

- On any set X we can pick the topology in which $\emptyset$ and X are the only open sets. If X has more than one point, then X is not Hausdorff.
- Given any polyhedron (like a cube) we define X to be the (finite) set of vertices, edges, and faces of the polyhedron. We define the closed subsets of X to be those unions of vertices, faces, and edges which are closed in the usual topology in $\boldsymbol{R}^3$. In the cube case, one of the closed sets has 9 elements: It is the union of a face and the 4 surrounding edges, and the 4 incident vertices. This weird topological space is a surprisingly geometric topology on a finite set. It is not Hausdorff.
- Let X denote the space $(\boldsymbol{R} - \{0\}) \cup 0_+ \cup 0_-$. As a set we get X by removing $\{0\}$ and replacing it with two "twin zeros". There are two natural subsets of X, namely $\boldsymbol{R}_+ = X - \{0_-\}$ and $\boldsymbol{R}_- = X - \{0_+\}$. We say that a basis for a topology on X is the set of open intervals in $\boldsymbol{R}_+$ and the set of open intervals in $\boldsymbol{R}_-$, when these two subsets are identified with $\boldsymbol{R}$. This turns X into a topological space. The inclusions $\boldsymbol{R}_\pm \to X$ are both continuous. (You just need to check this on the basis.) Note that X is not Hausdorff because you cannot place 0_+ and 0_- in disjoint open sets.

These few examples are just intended to show you the weirdness of non-Hausdorff topological spaces. This class will deal exclusively with Hausdorff topological spaces.

Compactness: A topological space X is called *compact* if every open cover of X has a finite sub-cover. You should recall the Heine-Borel Theorem: A subset of $\boldsymbol{R}^n$ with the usual topology is compact if and only if it is closed and bounded.

To say that a subset $K \subset X$ is compact is to say that K is a compact topological space when given the subspace topology. But, since open subsets of K are just intersections of K with open subsets of X, we get the following alternate characterization. A subset K of a topological space X is compact if every open cover of K by *open subsets of* X has a finite subcover.

Some Useful Lemmas: Here are some useful results about how the various properties we defined interact with each other.

Lemma 0.1 *A compact subset of a Hausdorff space is closed.*

Proof: It suffices to prove that the complement $X - K$ is open. For this, it suffices to prove that each $p \in X - K$ is contained in an open set U such that $U \subset X - K$.

Since X is Hausdorff, each $q \in K$ has the following property. There are open neighborhoods U_q and V_q such that $p \in U_p$ and $q \in V_q$ and $U_p \cap V_q = \emptyset$. Now $\{V_q\}$ is an open cover of K. Since K is compact, there is some finite subcover $V_1, ..., V_n$. Let $U = U_1 \cap ... \cap U_n$. By definition U is open, and by construction U is disjoint from $V_1 \cup ... \cup V_n$, a set which contains K. Hence $p \in U \subset X - K$. ♠

Lemma 0.2 *Suppose* $f : X \to Y$ *is a continuous map between topological spaces. If* X *is compact then* $f(X)$ *is compact.*

Proof: Let $\{U_\alpha\}$ be some open cover of $f(X)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X. Since f is continuous $f^{-1}(U)$ is open for each member U of this cover. Since X is compact, there is a finite subcover of X,

$$f^{-1}(U_1), ..., f^{-1}(U_n).$$

But then $U_1, ..., U_n$ is a cover of $f(X)$. Hence every open cover of $f(X)$ has a finite subcover. ♠

Lemma 0.3 *Suppose* X *is compact and* $K \subset X$ *is closed. Then* K *is compact.*

Proof: Choose an open cover $\{U_\alpha\}$ of K. Since K is closed, $V = X - K$ is open. But then $\{U_\alpha, V\}$ is an open cover of X. Since X is compact, there is a finite subcover. If this subcover contains V as a member, we delete V. In either case, we still have a finite cover of K. ♠

Lemma 0.4 *Suppose $f : X \to Y$ is a continuous bijection. If X is compact and Y is Hausdorff then f is a homeomorphism.*

Proof: The only property we need to check is the following one: If $U \subset X$ is open the $f(U)$ is open. This is equivalent to showing that if $K \subset X$ is closed then $f(K)$ is closed.

Since K is closed and X is compact, K is compact. But then $f(K)$ is compact. Since Y is Hausdorff, $f(K)$ is closed. ♠

Second Countability: A topological space X is *second-countable* if it has a countable basis. For instance $\boldsymbol{R}^n$ is second-countable. For a basis, you can use the open balls centered at rational points and having rational radii.

There are plenty of topological spaces which are not second-countable. For instance, we can put the *discrete topology* on an uncountable collection of points: Every set is open. This space is Hausdorff but not second-countable.

Topological Manifolds: A topological space X is an *n-dimensional topological manifold* if it has the following properties:

- X is Hausdorff.
- X is second-countable.
- Each point $p \in X$ is contained in an open subset U such that U is homeomorphic to $\boldsymbol{R}^n$.

Since open balls in $\boldsymbol{R}^n$ are homeomorphic to $\boldsymbol{R}^n$, you can replace the third condition by saying that U is homeomorphic to an open ball in $\boldsymbol{R}^n$.

It is a theorem, though not so easy to prove, that these conditions imply that X is homeomorphic to a metric space. So, if you don't like Hausdorff topological spaces, you can alternatively say that an n-dimensional topological manifold is a second-countable metric space that is locally homeomorphic to $\boldsymbol{R}^n$. This is kind of a weird definition, but it gives exactly the same objects.