M1720: Examples of Manifolds

These notes give some examples of manifolds. I will probably add to them over the next few weeks.

Open Subsets Any open subset of $U \subset \boldsymbol{R}^n$ is an n-dimensional smooth manifold. Each point $p \in U$ is contained in an open ball $B \subset U$, the pair (ι, B) serves as a coordinate chart. Here $\iota : B \to \boldsymbol{R}^n$ is the inclusion map. The overlap functions here are all restrictions of the identity map.

The Circle: Stereographic Projection One way to think of the circle S^1 is the solution of $x^2 + y^2 = 1$ in the plane. There is a specially nice map from $S^1 - (0,1)$ to $\boldsymbol{R}$. The map is given by

$$f_1(x,y) = \Big(\frac{x}{1-y}, 0\Big).$$

The inverse map is given by

$$f_1^{-1}(t,0) = \Big(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\Big).$$

Geometrically $(0,1)$ and (x,y) and $f_1(x,y)$ lie on the same ray emanating from $(0,1)$. The map f_1 is called *stereographic projection*, and it gives a coordinate chart from $S^1 - (0,1)$ to $\boldsymbol{R}$. The map

$$f_2(x,y) \to \Big(\frac{x}{1+y}, 0\Big)$$

gives a similar chart from $S^1 - (0,-1)$ to $\boldsymbol{R}$. The inverse map is given by

$$f_2^{-1}(t,0) = \Big(\frac{2t}{t^2+1}, \frac{1-t^2}{t^2+1}\Big).$$

We compute that

$$f_2 \circ f_1^{-1}(t) = 1/t.$$

So the overlap function is a diffeomorphism from $\boldsymbol{R} - \{0\}$ to itself. The two charts just defined give a smooth atlas for S^1, consisting of just two charts. This is great because then there is only one overlap function to consider and we have done it. This makes the circle into a smooth manifold.

The Circle: Quotient Space Approach:— One view of the circle is the quotient space $\boldsymbol{R}/\boldsymbol{Z}$, where two points $a, b \in \boldsymbol{R}$ are equivalent iff $a - b \in \boldsymbol{Z}$. Let $\pi : \boldsymbol{R} \to \boldsymbol{R}/\boldsymbol{Z}$ be the quotient map. Each point $p \in \boldsymbol{R}/\boldsymbol{Z}$ is contained in an open set U such that $\pi^{-1}(U)$ is a disjoint union of intervals, say of length $1/3$. We can choose any one of these intervals, say $\widetilde{U}$ and let $\phi : U \to \widetilde{U}$ be the map such that $\pi \circ \phi$ is the identity. In this case, all the overlap functions are translations. So, this is another way to see make S^1 into a smooth manifold. It is an easy but somewhat tedious exercise to show that the two methods give diffeomorphic manifolds.

The Sphere: We can think of S^n as the solution to the equation

$$\|\overrightarrow{x}\|^2 + y^2 = 1$$

where $\overrightarrow{x} = (x_1, ..., x_n)$. (This is just the usual equation if you expand it out.) There is a nice coordinate chart from $S^n - (\overrightarrow{0}, 1)$ to $\boldsymbol{R}^n$. This is the higher dimensional version of stereographic projection. The map is given by

$$f_1(\overrightarrow{x}, y) = \Big(\frac{\overrightarrow{x}}{1-y}, 0\Big).$$

The inverse is given by

$$f_1^{-1}(\overrightarrow{t}, 0\Big) = \Big(\frac{2\overrightarrow{t}}{\|\overrightarrow{t}\|^2 + 1}, \frac{\|\overrightarrow{t}\|^2 - 1}{\|\overrightarrow{t}\|^2 + 1}\Big)$$

This is the same equation as for the circle. There is a similar chart $f_2 : S^1 - (\overline{0}, -1) \to \boldsymbol{R}^n$ just as for the circle. The overlap function the map

$$f_2 \circ f_1^{-1} = \frac{\overrightarrow{t}}{\|\overrightarrow{t}\|^2}.$$

This agrees with the definition in the case of the circle. This is the most efficient way to make S^n into a smooth manifold.

The Torus: Quotient Space Approach On $\boldsymbol{R}^n$ we define the equivalence $p \sim q$ if $p - q \in \boldsymbol{Z}^n$. The quotient $\boldsymbol{R}^n/\boldsymbol{Z}^n$ is the n-torus. We have the projection map $\pi : \boldsymbol{R}^n \to \boldsymbol{R}^n/\boldsymbol{Z}^n$. Each point in $\boldsymbol{R}^n/\boldsymbol{Z}^n$ is the center of an open set U such that $\pi^{-1}(U)$ is an infinite union of disjoint open balls in $\boldsymbol{R}^n$.

We define $\phi : U \to \boldsymbol{R}^n$ by taking $\phi(U)$ to be some component of $\pi^{-1}(U)$ and forcing $\pi \circ \phi$ to be the identity map. With these coordinate charts, the overlap functions are all translations. This makes the n-torus into a smooth manifold.

The Torus: Cubes We can also think of the torus as the unit cube with the faces identified by translations. This works because the quotient space we get is exactly the same as the one we just described. That is, we get $\boldsymbol{R}^n/\boldsymbol{Z}^n$ again.

We can cut the big cube into 2^n cubes of side-length $1/2$ and then we can imagine suitably gluing these cubes together to make the torus. In the case of the 2-torus we would be making the 2-torus by gluing together 4 squares in an appropriate pattern. You should try drawing this out.

Linear Fractional Transformations: As a prelude to talking about higher genus surfaces, we discuss linear fractional transformations. A *linear fractional transformation* is a map of the form

$$z \to \frac{az+b}{cz+d},$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$. The domain for this map is $\boldsymbol{C} \cup \infty$. The image of ∞ is defined to be a/c.

Let me give some information about these maps. As you can see by directly solving the equations, there is a unique linear fractional transformation T such that $T(z_i) = w_i$ for $i = 1, 2, 3$. Here z_1, z_2, z_3 are distinct points and so are w_1, w_2, w_3. A *circle* in $\boldsymbol{C} \cup \infty$ is either a round circle or else a union $L \cup \infty$ where L is a straight line.

Lemma 0.1 *Linear fractional transformations map circles to circles.*

Proof: You could just compute this out, but it is a bit painful. Here is a conceptual proof. Let C be an arbitrary circle. Call a linear fractional transformation T *good* if $T(C)$ is a circle. We want to prove that all linear fractional transformations are good.

The map

$$T_0(z) = \frac{z-i}{z+i}$$

maps $\boldsymbol{R} \cup \infty$ to the unit circle. Composing T_0 with maps of the form

$$z \to az + b,$$

a special kind of linear fractional transformation, we see that there is a linear fractional transformation which maps $\boldsymbol{R} \cup \infty$ to any circle we like. Using the fact that the linear fractional transformations form a group, we see that for any circle C' there is a linear fractional transformation L such that $L(C) = C'$. We get this linear transformation by mapping C to $\boldsymbol{R} \cup \infty$ and then mapping $\boldsymbol{R} \cup \infty$ to C'.

Now, we can do this procedure in many ways. After we map $\boldsymbol{C}$ to $\boldsymbol{R} \cup \infty$ we can apply a self-map of $\boldsymbol{R} \cup \infty$ that maps any three points on $\boldsymbol{R} \cup \infty$ to any other three points. This means that for any triple of points (z_1, z_2, z_3) on C and any triple (z_1', z_2', z_3') on C' we can find a linear transformation L such that $L(C) = C'$ <u>and</u> $L(z_i) = z_i'$ for $i = 1, 2, 3$.

If some linear fractional transformation F is bad, and $F(C)$ is not a circle, then we pick three points $z_1, z_2, z_3 \in C$ and look at their images $z_i' = F(z_i)$. There is some circle C' containing z_1', z_1', z_3'. From what we have said above, we must have $F(C) = C'$. Hence F is good after all. ♠

Hyperbolic Geometry Prelude: As a prelude to talking about higher genus surfaces, we talk about the hyperbolic plane. A model for the hyperbolic plane is the open unit disk, which we denote by $\boldsymbol{H}^2$. We denote the boundary of $\boldsymbol{H}^2$ by S^1. The *geodesics* in $\boldsymbol{H}^2$ are arcs of circles which meet the boundary at right angles.

An *orientation preserving automorphism* of $\boldsymbol{H}^2$ is a linear fractional transformation which maps $\boldsymbol{H}^2$ to itself in a bijective way. There are lots of these: We just pick two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of clockwise oriented points in S^1 and then take the linear fractional transformation T such that $T(z_i) = w_i$ for $i = 1, 2, 3$. Linear fractional transformations also preserve angles, because they map circles to circles. So, the derivative is a similarity. (For those of you who know complex analysis, this means that linear fractional transformations satisfy the Cauchy-Riemann equations.) From this property we see that an automorphism of $\boldsymbol{H}^2$ maps geodesics to geodesics.

The map $z \to \overline{z}$ also preserves $\boldsymbol{H}^2$ and maps geodesics to geodesics and preserves angles. An *automorphism* of $\boldsymbol{H}^2$ is any map of $\boldsymbol{H}^2$ which is finite composition of linear fractional transformations and the map $z \to \overline{z}$.

It turns out that we can make $\boldsymbol{H}^2$ into a metric space in such a way that all automorphisms are isometries. Given $b, c \in \boldsymbol{H}^2$ we let $a, d \in S^1$ be the points so that a, b, c, d appear on the geodesic through b and c. The distance from b to c is given by

$$\log \frac{(a-c)(b-d)}{(a-b)(c-d)}.$$

You can check by a direct calculation that this is invariant. It is a bit harder to show that this also satisfies the triangle inequality, but this is not something we need to know. We mention this because the automorphisms of $\boldsymbol{H}^2$ are precisely the isometries of this space. $\boldsymbol{H}^2$ is very much like the Euclidean plane; it is homogeneous and isotropic.

A *geodesic polygon* in $\boldsymbol{H}^2$ is a region bounded by a finite chain of geodesic segments. If you draw a few figures, you can see that it is possible to make a right angled and totally symmetric geodesic hexagon. (Small hexagons will have angles close to their Euclidean counterparts and big ones will have angles near 0.) Call this thing a *regular right-angled hexagon*, or *RRH* for short. The image of a regular right-angled hexagon under an automorphism of $\boldsymbol{H}^2$ is also called an *RRH*. All the sides of an RRH have the same length, and all the angles are right angles.

Higher Genus Surfaces: Take two copies of an RRH and color the sides alternately red and glue on each one. Now glue the blue sides of one to the blue sides of the other by the identity map. Topologically, this produces a sphere with 3 holes cut out. This is how you might sew a pair of pants together.

Now take two pairs of pants. Each one has 3 red circular boundary components. Glue each boundary component on one to the corresponding boundary component on the other. Topologically, you have made a genus 2 surface.

To get a nice atlas of coordinate charts, we can map the neighborhood of each point of the quotient space into $\boldsymbol{H}^2$ by maps which are local isometies on each hexagon. If you do this, the overlap functions will be automorphisms of $\boldsymbol{H}^2$ and hence smooth. Just as you can build the torus out of 4 squares, you can build the genus 2 surface out of 4 RRGs.

You can build the genus 3 surface out of 8 RRGs. And so on.