Math 181 Handout 2

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The purpose of this handout is to explain some methods for producing surfaces and also higher dimensional manifolds. You can safely skip the material on manifolds if you want.

0.1 What is a Manifold?

Recall that a surface is a metric space, such that every point has a neigborhood which is homeomorphic to $\boldsymbol{R}^2$. Motivated by this, we have

Definition: An *n-dimensional manifold* is a metric space, such that every point has a neighborhood which is homeomorphic to $\boldsymbol{R}^n$.

So, you can see that a manifold is really a straightforward generalization of surfaces to higher dimensions. Our definition allows for the possibility that the empty set is an n-dimensional manifold. You can either accept this triviality or reject it as you see fit.

Technical Comment: This definition of a manifold is slightly nonstandard. The usual definition replaces *metric space* with *Hausdorff topological space*. However, in most cases the metric space definition coincides with the topological space definition−i.e. singles out the same objects as manifolds. The reason I'm using the metric space definition is that it's more concrete. If you want to know about topological spaces, Hausdorff or otherwise, ask me in office hours.

1 Coordinate Patches

One simple way to get a surface is to take the graph of a continuous function $f : U \to \boldsymbol{R}$. Here $U \subset \boldsymbol{R}^2$ is an open set. You can form the subset of $\boldsymbol{R}^3$

$$\Gamma(U, f) = \{(x, y, f(x, y))|\ (x, y) \in U\}.$$

This is just the graph of f over U. The set $\Gamma(U, f)$ inherits a metric from $\boldsymbol{R}^3$.

Exercise 1: Prove that $\Gamma(U, f)$ is a surface.

Of course you could do the same thing in higher dimensions, to produce examples of higher dimensional manifolds.

2 Hypersurfaces

2.1 The Basic Theorem

Now let $f : \boldsymbol{R}^3 \to \boldsymbol{R}$ be a continuous function. Assume also that the partial derivatives of f exist and are continuous functions. This means that the gradient

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f)$$

exists and is continuous. Say that 0 is a *regular value* for f if it never happens that both $f(x, y, z) = 0$ and $\nabla f(x, y, z) = (0, 0, 0)$ (at the same point.)

Theorem 2.1 *If* 0 *is a regular value for* f *then* $f^{-1}(0)$ *is a surface provided that it is nonempty.*

For instance, 0 is a regular value for the function $f(x, y, z) = x^2+y^2+z^2-1$ and the set $f^{-1}(0)$ is the sphere. This is probably the hardest way to prove that the sphere is a surface!

A similar theorem holds if you replace $\boldsymbol{R}^3$ by $\boldsymbol{R}^n$. A nice higher dimensional example is obtained as follows. You can think of the set of 2×2 (real valued) matrices as a copy of $\boldsymbol{R}^4$. There is a nice map from this space into $\boldsymbol{R}$, namely the determinant (minus 1):

$$f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc - 1.$$

Exercise 2: Show that 0 is a regular value for f.

Technical Comments: The set $f^{-1}(0)$ is usually denoted by $SL_2(\boldsymbol{R})$. Thus $SL_2(\boldsymbol{R})$ is the set of unit determinant real 2×2 matrices. Combining Exercise 2 with the theorem above, we see that $SL_2(\boldsymbol{R})$ is a (3-dimensional) manifold. If you know about groups then you probably know that $SL_2(\boldsymbol{R})$ forms a group under matrix composition. Thus $SL_2(\boldsymbol{R})$ is both a manifold and a group−and it turns out that the two structures are "compatible" in a way that I can explain in office hours. Objects with these two structures (coexisting in a compatible way) are called *Lie groups*, and $SL_2(\boldsymbol{R})$ is one of the most important examples. A similar game works for $SL_n(\boldsymbol{R})$, the "group/manifold" of determinant 1 real $n \times n$ matrices.

OK, back to earth. Theorem 2.1 is really a special case of the Implicit Function Theorem. However, you don't need to know the I.F.T. in order to understand the self-contained proof I'll give below. I'm only going to give the proof for surfaces, but if you understand this case you should see how the proof generalizes.

2.2 Proof of Theorem 2.1

Let $S = f^{-1}(0)$. Let $p = (x, y, z) \in S$. We want to show that p has a neighborhood $U \subset S$ which is homeomorphic to $\boldsymbol{R}^2$. We know that $\nabla f(p)$ is nonzero, so there is a unique plane P such that $p \in P$ and $\nabla f(p)$ is perpendicular to P. Without loss of generality we can rotate and translate space, and replace f by a scalar multiple Cf to arrange that

- $p = (0, 0, 0)$.
- $\nabla(p) = (0, 0, 1)$.

In this case P is the xy plane.

Let Q_ϵ denote the open cube of diameter ϵ centered at 0. If ϵ is sufficiently small then

$$\|\nabla f(q) - (0, 0, 1)\| < \frac{1}{1000000},$$

for all $q \in Q_\epsilon$. In other words, the gradient almost points straight up throughout Q_ϵ. (This is really overkill; we don't need 1/1000000, but it makes things more obvious to take a really tiny constant like this.) Let's take such a choice of ϵ and write $Q = Q_\epsilon$.

Let $U = Q \cap S$. Then U is an open neighborhood of p in S. We just have to show that U is homeomorphic to $\boldsymbol{R}^2$. It suffices to show that U is homeomorphic to an open square, since an open square is homeomorphic to $\boldsymbol{R}^2$. As it happens, $Q \cap P$ is an open square, and the map

$$h(x, y, z) = (x, y, 0).$$

Is a map from U to $Q \cap P$. We just have to show that h is a homeomorphism. Here are the main points:

- h is a distance decreasing map so (using the $\epsilon - \delta$ definition of continuity) h is continuous.

- To show that h is one-to-one, suppose that $h(q_1) = h(q_2)$ for some points $q_1, q_2 \in U$. But then q_1 and q_2 lie on the same vertical line. Here comes the key point. Say that q_1 lies below q_2. But then the upward vertical path from q_1 to q_2 at all locations nearly points in the same direction as ∇f. In other words, $\partial_z f > 0$ along the vertical path joining q_1 to q_2. But then $f(q_2) > f(q_1)$. This contradicts the fact that $f(q_2) = f(q_1) = 0$. This contradiction shows that h is one to one.

- **Exercise 3A:** Show that h is onto. Hint: Show that f is negative on the bottom face of Q and positive on the top face. Then f has to be zero on each vertical line segment which connects the top and bottom faces.

- **Exercise 3B:** Show that h^{-1} is continuous. Hint: Suppose that (x_1, y_1) and $(x_2, y_2) \in Q \cap P$ are very close together. Consider $(x_1, y_1, z_1) = h^{-1}(x_1, y_1)$ and $(x_2, y_2, z_2) = h^{-1}(x_2, y_2)$. Suppose that z_1 and z_2 are far apart and derive a contradiction by looking at the directional derivative along the near vertical line segment joining (x_1, y_1, z_1) to (x_2, y_2, z_2).

These items show that h is a homeomorphism from U to the open square $Q \cap P$. Since p was an arbitrary point, we've shown that every point on S has a neighborhood which is homeomorphic to $\boldsymbol{R}^2$.

2.3 A Generalization

There is a generalization of the above theorem, which works when you have a map $f : \boldsymbol{R}^n \to \boldsymbol{R}^k$, with $k < n$. In this case we require that the differential map df is everywhere defined and continuous. The map df is just the matrix of partial derivatives. At each point it is a linear map from $\boldsymbol{R}^n$ to $\boldsymbol{R}^k$. (When $k = 1$ the map df is just the gradient.)

The point $p \in \boldsymbol{R}^n$ is said to be a *regular point* for f if the map $df(p)$ is onto. This is to say that the matrix of partials has full rank. We say that 0 is a *regular value* for f if f is regular at every point of $f^{-1}(0)$. In this case, the result is that $f^{-1}(0)$ is an $(n-k)$ dimensional manifold. This result again is an application of the I.F.T., though one can give a self-contained proof along the lines of what I did in the special case that $k = 1$.

3 Gluing Spaces Together

Now I'm going to describe a general construction which is usually done for topological spaces. However, it can be done for metric spaces as long as we're a bit careful. The advantage to using topological spaces is that the construction always works. The disadvantage to using topological spaces is that it takes a long time to figure out what the construction actually means. For metric spaces, things don't always work out, but whatever happens is more understandable. Also, for our purposes, things always work out.

3.1 A Word about the Reals

Before we start we need to recall the notion of the *inf* from real analysis. Let $S \subset \boldsymbol{R}$ be a set consisting entirely of non-negative numbers. Then $x = \inf S$ denotes the smallest member of the closure of S. Such a number always exists and is unique. The existence (and uniqueness) of the inf is known as the *completeness axiom* for the reals.

3.2 Trianglizations

Let X be a set and let $\delta : X \times X \to \boldsymbol{R}$ be a map which just satisfies $\delta(x,y) = \delta(y,x) \geq 0$. Note that δ need not satisfy the triangle inequality. The purpose of this section is to show how to replace δ by a new function which sometimes remembers some of the structure of δ and yet satisfies the triangle inequality.

Let $x, y \in X$ be two points. Say that a *chain* from x to y is a finite sequence of points $x = x_0, x_1, ..., x_n = y$. Let's call this chain C. Let's define

$$\delta(C) = \delta(x_0, x_1) + \delta(x_1, x_2) + ... + \delta(x_{n-1}, x_n).$$

Certainly $\delta(C) \geq 0$ as long as $x \neq y$. Now lets define

$$d(x, y) = \inf_C \delta(C).$$

The inf is taken over the set of all possible values $\delta(C)$ where C is a chain from x to y.

This probably looks like an insane definition, but let's try to make it intuitive. Think of $\delta(x, y)$ as the cost of flying from city x to city y—let's say from Providence to Tahiti. Now, you're really desperate to get to Tahiti, and have tons of free time but little money. So, you look on the internet and try to find all possible flights. You are willing to take any conceivable chain of connecting flights, as long as you start in Providence and end in Tahiti. After searching through all the possiblities you select the most economical flight. This is $d(x, y)$. The difference between this scenario and the idealized one we're talking about is that X could be an infinite metric space. So, there could be infinitely many chains, and you need to take the inf rather than just a minimum (which may not exist.) We call d the *trianglization* of δ because we construct d in such a way to force it to satisfy the triangle inequality. (This is my own term. I don't know what this thing is called by other people.)

Exercise 4: Show that d satisfies the following axioms:

- $d(x, y) \geq 0$.
- $d(x, y) = d(y, x)$.
- $d(x, y) \leq d(x, z) + d(z, y)$.

So it looks like d is a metric. However, note the that we're leaving off the part that would say $d(x, y) = 0$ iff $x = y$. In fact give an example of a δ on $X = \boldsymbol{R}^2$, which satisfies the first two axioms for a metric, whose trianglization is the zero map.

3.3 The Quotient Construction

Let X be a set. An *equivalence relation* on X is a relation of the form $\sim$, which satisfies three properties:

- $x \sim x$ for all x.
- $x \sim y$ iff $y \sim x$.
- $x \sim y$ and $y \sim z$ imply $x \sim z$.

An *equivalence class* is a subset

$$S = \{y \in X |\ y \sim x\}.$$

So, S is the set of all elements which are equivalent to x. Note that every two equivalence classes are either disjoint or identical. Thus, it makes sense to talk about the set of equivalence classes. This set is denoted $X/\sim$.

Now let's see how $\sim$ interacts with a metric. Let d' be a metric on X. As above, let $X/\sim$ denote the set of equivalence classes of X. Let's define, for $S_1, S_2 \in [X]$, the function

$$\delta(S_1, S_2) = \inf d'(s_1, s_2).$$

The inf is taken over all possibilities where $s_1 \in S_1$ and $s_2 \in S_2$. In other words the "distance" from S_1 to S_2 is the "minimum" distance between a member of S_1 and a member of S_2. The "minimum" doesn't always make sense in this context and so we use the inf.

Let d be the trianglization of δ. We call $X/\sim$ a *good quotient* if d is a metric on $X/\sim$.

Exercise 5: Let $X = \boldsymbol{R}$ and write $x = y$ iff $x - y$ is rational. Show that $\boldsymbol{R}/\sim$ is not a good quotient.

Exercise 6: (The Asteroids Exercise) This is a very important exercise for this class: On $\boldsymbol{R}^2$ define $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 - x_2$ and $y_1 - y_2$ are both integers. Prove that $X/\sim$ is a good quotient, and the resulting metric space is a surface homeomorphic to the surface of a donut!

3.4 Examples

Let's give some examples of the abstract constructions considered above. Let $X = X_1 \cup X_2$ where X_1 and X_2 are each copies of the unit disk, equipped with the standard metric, and $d(p_1, p_2) = 1$ if $p_1 \in X_1$ and $p_2 \in X_2$. You should picture two disks hovering, one on top of the other. Define $p_1 \sim p_2$ if and only if either $p_1 = p_2$ or else p_1 and p_2 are corresponding points in the boundaries of X_1 and X_2.

Exercise 7: Prove that the space $X/\sim$ is a good quotient, and is homeomorphic to the 2-sphere.

Exercise 8: Let $X = S^1 \times [0, 1]$ be a cylinder. Define an equivalence relation by the rule that $(x, 0) \sim (x, 1)$ and also $(x, y) \sim (x, y)$. Prove that $X/\sim$ is a good quotient, and also a surface, and also homeomorphic to the space in the Asteroids problem.

Exercise 9: Let X be a metric space of the form $T \times \{1, 2, 3, 4, 5, 6, 7, 8\}$. So, X is the disjoint union of 8 triangles. Define an equivalence relation on X so that the resulting space is a surface and homeomorphic to a sphere.

The operation we have been doing is sometimes called *gluing*. The idea is that we take distinct points on the space and then call them *equivalent*. The process of taking the quotient is sort of like gluing the spaces together along these points because it declares two equivalent points the same, so that they really are glued together. Moreover, if $x \sim y$ and x' is near x and y' is near y, then the trianglization process forces x' to be near y'. So, when we glue two equivalent points together, we sort of drag the rest of the space with us. This is what you would actually experience if you tried, say, to glue together parts of a rubber sheet.

You can probably see from the previous two exercises that it is possible to build up more complicated surfaces from gluing together simpler pieces.

Exercise 10: (Challenge) Can you glue a finite number of triangles together (as in Exercise 9) to produce a surface which is neither homeomorphic to a sphere or to a torus? (Hint: first draw some candidate surfaces and then see how to break them apart into triangles.)