Math 181 Handout 5

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October 2, 2005

The purpose of this handout is to define the notions of covering space and deck transformation group, and to relate them to the fundamental group. The handout also contains a swift "review" of some real analysis.

1 The Bolzano-Weierstrass Theorem

Before we get started, we need to recall a bit of real analysis. You can find this material in any book on the subject−e.g Steven R. Lay's book, *Analysis with an Introduction to Proof*.

A sequence of points $\{c_j\}$ in a metric space X is called *Cauchy* if, for every $\epsilon > 0$, there is some N such that $i, j > N$ implies that $d(c_i, c_j) < \epsilon$. A convergent sequence is automatically Cauchy, and one can ask about the converse. X is said to be *complete* if every Cauchy sequence in X converges to a point in x.

Exercise 1: Prove that $\boldsymbol{Q}$, the space of rationals, is not complete.

The basic axiom for $\boldsymbol{R}$ is that it is complete. You might ask how one proves that $\boldsymbol{R}$ is complete. One way to do this is to deduce it from the fact that every non-negative subset of $\boldsymbol{R}$ has an inf. (This is the *greatest lower bound property*.) Then you can ask how to prove that $\boldsymbol{R}$ has the greatest lower bound property. The usual approach is to *construct* $\boldsymbol{R}$ from $\boldsymbol{Q}$ in such a way that the greatest lower bound property holds. Alternatively, you can construct $\boldsymbol{R}$ from $\boldsymbol{Q}$ in such a way that it is complete. If you are interested in the construction of $\boldsymbol{R}$ from $\boldsymbol{Q}$ (and you should be!) ask me about it.

Exercise 2: Using the completeness of $\boldsymbol{R}$ as an axiom prove the following result: Let $Q_1 \supset Q_2 \supset Q_3 ...$ be a nested sequence of cubes in $\boldsymbol{R}^n$ such that the diameter of Q_n tends to 0 with n. Then $\bigcap Q_n$ is one point. (Hint: look at the sequence of centers.)

Theorem 1.1 (Bolzano-Weierstrass) *A sequence $\{c_n\}$ contained in the unit cube Q_0 has a convergent subsequence.*

Proof: Note that Q_0 is the union of 2^n cubes having half the size. One of these subcubes Q_1 must contain c_j for infinitely many indices. But Q_1 is the union of 2^n cubes having half the size. One of these subcubes Q_2 must contain c_j for infinitely many indices. Any so on. The intersection $\bigcap Q_j$, a single point by Exercise 2, is the limit of some subsequence of $\{c_j\}$. ♠

2 Covering Spaces

Let $\widetilde{X}$ and X be path connected metric spaces. Let $E : \widetilde{X} \to X$ be a continuous map. An open set $U \in X$ is said to be *evenly covered* if the preimage $E^{-1}(U)$ consists of a countable disjoint union of sets $\widetilde{U}_1, \widetilde{U}_2, ...$ such that the restriction $E : \widetilde{U}_j \to U$ is a homeomorphism. (This makes sense because $\widetilde{U}_j$ is a metric space in its own right.) It is customary to require that U is path connected. The sets $\widetilde{U}_j$ are called *components* of the preimage.

The map E is said to be a *covering map* if every point in X has a neighborhood which is evenly covered. In this case $\widetilde{X}$ is said to be a *covering space* of X. Some examples:

- The mother of all examples is $E : \boldsymbol{R} \to S^1$, where $E(x) = \exp(2\pi i x)$.
- **Exercise 3:** Let S^2 be the 2 sphere and let $\boldsymbol{P}^2$ be the projective plane, defined as the set of equivalence classes of antipodal points on S^2. Show that the obvious map $S^2 \to \boldsymbol{P}^2$ is a covering map. (Note: in order to do this problem you have to recall the metric on $\boldsymbol{P}^2$.)
- **Exercise 4:** Let θ be the graph which is homeomorphic to the letter θ. Let T_3 be the 3-valent infinite tree. Exhibit a map $E : T_3 \to \theta$ which is a covering map. (Note, θ is very similar to the figure 8 graph we considered in class.)

3 The Lifting Property

In this section $E : \widetilde{X} \to X$ is a covering map.

Let Q be a cube and let $f : Q \to X$ be a continuous map. We say that a *lift* of f is a map $\widetilde{f} : Q \to \widetilde{X}$ such that $E \circ \widetilde{f} = f$. This notion is just the generalization of what we talked about in the previous handout. The purpose of this section is to prove the formal version of the result we talked about, for some examples, in the previous handout.

We begin with a technical result.

Lemma 3.1 *There is some N with the following property. If $Q' \subset Q$ is a sub-cube with side length less than $1/N$ then $f(Q')$ is contained in an evenly covered neighborhood of X.*

Proof: If this result is false then we can find a sequence of sub-cubes $\{Q_j\}$, with the diameter tending to 0, such that $f(Q_j)$ is not contained in an evenly covered neighborhood. Let c_j be the center of Q_j. Then the sequence $\{c_j\}$ has a convergent subsequence. Tossing out everything but the cubes corresponding to this subsequence we can assume that $\{c_j\}$ is a convergent sequence. Let x be the limit point, guaranteed by the Bolzano-Weierstrass Theorem. Then $f(x)$ is contained in an evenly covered neighborhood $U \subset X$. But then $f(Q_n) \subset U$ for n large, by continuity. This is a contradiction. ♠

Lemma 3.2 *Let Q be a cube and let $f : Q \to X$ be a continuous map. Let v be a vertex of Q and let $\widetilde{x} \in X$ be a point such that $E(\widetilde{x}) = f(v)$. Suppose that $f(Q)$ is contained in an evenly covered neighborhood. Then there is a unique lift $\widetilde{f} : Q \to \widetilde{X}$ such that $\widetilde{f}(v) = \widetilde{x}$.*

Proof: Let $U \subset X$ be the evenly covered neighborhood such that $f(Q) \subset U$. Recall that $E^{-1}(U)$ is a disjoint union of sets $\widetilde{U}_1, \widetilde{U}_2, ...$ such that the restriction $E : \widetilde{U}_j \to U$ is a homeomorphism. Let $\widetilde{U}_k$ be the component which contains $\widetilde{x}$. Let F be the inverse of the restriction E to $\widetilde{U}_k$. Then we can and must define $\widetilde{f} = F \circ f$. ♠

Just as we did in the previous handout we want to not remove the hypothesis that $f(Q)$ is contained in an evenly covered neighborhood.

Theorem 3.3 *Let Q be a cube and let $f : Q \to X$ be a continuous map. Let v be a vertex of Q and let $\widetilde{x} \in X$ be a point such that $E(\widetilde{x}) = f(v)$. Then there is a unique lift $\widetilde{f} : Q \to \widetilde{X}$ such that $\widetilde{f}(v) = \widetilde{x}$.*

Proof: By Lemma 3.1 we can find some N such that any subcube of Q of diameter less than N is mapped into an evenly covered neighborhood by f. Let's partition Q into such cubes, say $Q = Q_1, ..., Q_m$. We can order these cubes so that, for each k, the cube Q_k shares a vertex v_k with some Q_j for $j = 1, ..., k-1$. Also we set things up to that the initial vertex $v = v_1$ is a vertex of Q_1. We define $\widetilde{f}$ on Q_1 using Lemma 3.2. This tells us the value of $\widetilde{f}$ on v_2 and lets us define $\widetilde{f}$ on Q_2. The uniqueness guarantees that the definition on Q_2 is compatible with the definition on Q_1. And so on. When we are done, we have defined $\widetilde{f}$ in the only way possible on all of Q. ♠

We will only need this result for the case of the unit interval $[0, 1]$ and the unit square $[0, 1]^2$, but is it nice to know in general.

4 The Deck Group

We've already associated one group to a (pointed) metric space, namely the fundamental group. Now we are going to assign a group in a second way. Let $E : \widetilde{X} \to X$ be a covering map as above. Say that a *deck transformation* is a homeomorphism $h : \widetilde{X} \to \widetilde{X}$ such that $E \circ h = E$.

An explanation of the name actually gives some insight into what these things are. Suppose you have a deck of cards. There is a natural map E, from your deck of cards, to a single card. (You can think of holding the deck directly above the single card and then E is vertical projection.) Now, if you shuffle the cards and re-do the map E there is no change. So, a deck transformation in this case corresponds to shuffling the deck.

In general, you can think of $\widetilde{X}$ as a kind of deck of cards and X as a single card. The analogy isn't perfect because $\widetilde{X}$ is connected, but for an evenly covered neighborhood $U \subset X$ the set $\widetilde{U} = E^{-1}(U)$ really is like a deck of cards. The deck transformation h somehow permutes the disjoint components of $\widetilde{U}$ like shuffling permutes the cards.

If h is a deck transformation, so is h^{-1}. Likewise, if h_1 and h_2 are deck transformations, so is $h_1 \circ h_2$. Thus the set of deck transformations forms a group under composition. This group is called the deck group of $(\widetilde{X}, X, E)$.

Exercise 5: Let X be the figure 8 space and let $\widetilde{X}$ be the infinite 4-valent tree. Let E be the covering map discussed in class. Prove that the deck group for $(\widetilde{X}, X, E)$ is isomorphic to the free group F_2 on two letters.

In this example, the deck group turns out to be isomorphic to the fundamental group of the figure 8 space. This is not an accident.

5 Simply Connected Spaces

Let X be a path connected metric space. X is said to be *simply connected* if $\pi_1(X)$ is trivial. This definition does not depend on the basepoint, because the isomorphism type of the fundamental group is indepdendent of basepoint in path connected spaces.

Suppose that $f_0, f_1 : [0,1] \to X$ are two paths. Suppose also that

$$f_0(0) = f_1(0); \qquad f_1(0) = f_1(1).$$

In other words, the two paths have the same beginning and the same ending. We say that f_0 and f_1 are *path homotopic* if there is a homotopy F from f_0 to f_1 such that $f_t(0) = f_0(0)$ and $f_t(1) = f_0(1)$ for all t. Here, as usual, $f_t(x) = F(x,t)$. In other words, all the paths f_t start and end at the same points as do the paths f_0 and f_1. Intuitively, a path homotopy slides f_0 to f_1 without moving the endpoints.

In the case that $f_0(0) = f_0(1) = f_1(0) = f_1(1)$ the notion of a path homotopy coincides with the notation of a loop homotopy.

Exercise 6: Suppose that X is simply connected. Prove that any two paths, which have the same endpoints as each other, are path homotopic. Outline: Let x be the starting point of both loops. Consider the loop g formed by first doing f_0 and then doing f_1. Then $[g] \in \pi_1(X,x)$. Hence g is loop homotopic to the identity. Let G be this loop homotopy. Try to modify G slightly so that G becomes a path homotopy from f_0 to f_1.

6 The Isomorphism Theorem

Here is the main theorem of this handout, and (in my opinion) one of the best theorems in algebraic topology:

Theorem 6.1 *Suppose that*

- $E : \widetilde{X} \to X$ *is a covering map.*
- X *and* $\widetilde{X}$ *are path connected.*
- $\widetilde{X}$ *is simply connected.*

Then $\pi_1(X)$ *is isomorphic to the deck group for* $(\widetilde{X}, X, E)$.

The rest of this handout is devoted to the proof.

6.1 Step 1: Define the Isomorphism

Since X is path connected $\pi_1(X, x)$ is independent of basepoint. Let $x \in X$ be a basepoint. Let $G = \pi_1(X, x)$. Let D be the deck transformation group. Here we will define a map $\Phi : D \to G$. In later steps we will show that Φ is an isomorphism.

Let $\widetilde{x} \in \widetilde{X}$ be some point such that $E(\widetilde{x}) = x$. We make this choice once and for all. Suppose that $h \in D$ is a deck transformation. Then $\widetilde{y} = h(\widetilde{x})$ is some other point. Note that

$$E(\widetilde{y}) = E(h(\widetilde{x})) = E(\widetilde{x}) = x.$$

Since $\widetilde{X}$ is path connected, there is some path $\widetilde{f} : [0, 1] \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x}$ and $\widetilde{f}(1) = \widetilde{y}$. Let $f = E \circ \widetilde{f}$. By construction $f(0) = f(1) = x$. Hence f is a loop based at x. Define

$$\Phi(h) = [f] \in G.$$

To see that Φ is well defined. Suppose that f_0 and f_1 are two paths connecting $\widetilde{x}$ to $\widetilde{y}$. Since $\widetilde{X}$ is simply connected, there is a path homotopy $\widetilde{F}$ from $\widetilde{f}_0$ to $\widetilde{f}_1$. But then $F = E \circ \widetilde{F}$ is a loop homotopy from f_0 to f_1. Hence $[f_0] = [f_1]$ and Φ is well defined.

6.2 Homomorphism

This step looks quite mysterious, but is fairly obvious if you draw pictures.

Let $h_1, h_2 \in D$ be two deck transformations. We want to show that

$$\Phi(h_1 \circ h_2) = \Phi(h_1)\Phi(h_2).$$

Let $\widetilde{y}_j = h_j(\widetilde{x})$ for $j = 1, 2$. Let f_j be a path in $\widetilde{X}$ joining $\widetilde{x}$ to $\widetilde{y}_j$. Let $f_j = E \circ \widetilde{f}_j$. Then $\Phi(h_j) = [f_j]$, as above.

Let $\widetilde{z} = h_1 \circ h_2(\widetilde{x}_1)$. Note that $h_1 \circ \widetilde{f}_2$ is a path joining

$$h_1(\widetilde{x}) = \widetilde{y}_1$$

to

$$h_1(\widetilde{y}_2) = h_1 \circ h_2(\widetilde{x}) = \widetilde{z}.$$

Therefore, the concatenated path $\widetilde{f}_1 * (h_1 \circ \widetilde{f}_2)$ joins $\widetilde{x}$ to $\widetilde{z}$. But then

$$\Phi(h_1 \circ h_2) = [E \circ (\widetilde{f}_1 * (h_1 \circ \widetilde{f}_2))] = [(E \circ \widetilde{f}_1) * (E \circ h_1 \circ \widetilde{f}_2)] =^*$$

$$[(E \circ \widetilde{f}_1) * (E \circ \widetilde{f}_2)] = [f_1 * f_2] = [f_1][f_2] = \Phi(h_1)\Phi(h_2).$$

The starred equality comes from the fact that $E \circ h_1 = E$.

Exercise 7: Pick a nice example, e.g. $\widetilde{X} = \boldsymbol{R}^2$ and $X = T^2$, the torus, and go through the above argument step by step, illustrating the proof with pictures.

6.3 Injectivity

Since Φ is a homomorphism we can show that Φ is injective just by showing that the kernel of Φ is trivial. So, suppose that $\Phi(h)$ is the trivial element in $\pi_1(X, x)$.

Lemma 6.2 $h(\widetilde{x}) = \widetilde{x}$.

Proof: Let $\widetilde{y} = h(\widetilde{x})$. We want to show that $\widetilde{y} = \widetilde{x}$. Let $\widetilde{f}$ be a path which joins $\widetilde{x}$ to $\widetilde{y}$. It suffices to show that $\widetilde{f}$ is path homotopic to the constant path. This is what we will do.

Let $f = E \circ \widetilde{f}$. Then $\Phi(h) = [f]$. By hypotheses, there is a loop homotopy F from f to the trivial loop. Let Q be the unit square. By construction $F : Q \to X$ is a continuous map such that $f_0 = f$ and f_1 is the constant map. From the lifting theorem there is a lift $\widetilde{F} : Q \to \widetilde{X}$ such that $\widetilde{F}(0, 0) = \widetilde{x}$ and $E \circ \widetilde{F} = F$. Here are some properties of $\widetilde{F}$:

- $\widetilde{f}_0$ is a lift of $f_0 = f$. From the uniqueness of lifts, $\widetilde{f}_0 = \widetilde{f}$.
- $\widetilde{f}_1$ is the constant path since f_1 is the constant path.

- $F(0,t)$ and $F(1,t)$ are the basepoint in X, independent of t. Therefore $\widetilde{F}(0,t)$ and $\widetilde{F}(1,t)$ are constant maps. In other words, the endpoints of $\widetilde{f}_t$ do not change with t. That is, $\widetilde{F}$ is a path homotopy from our path $\widetilde{f}$ to the constant path.

The last item completes our proof. ♠

Lemma 6.3 *If $h(\widetilde{x}) = \widetilde{x}$ then h is the identity map.*

Proof: Let $\widetilde{y}$ be some other point in $\widetilde{X}$. We want to show that $h(\widetilde{y}) = \widetilde{y}$. Let $\widetilde{f}$ be a path joining $\widetilde{x}$ to $\widetilde{y}$. Let $x = E(\widetilde{x})$ and $y = E(\widetilde{y})$. Let $f = E \circ \widetilde{f}$. Then $f : [0,1] \to X$ is a path which joins x to y.

The paths $\widetilde{f}$ and $h \circ \widetilde{f}$ are both lifts of f which agree at the point 0. That is $\widetilde{f}(0) = \widetilde{x}$ and $h \circ \widetilde{f}(0) = h(\widetilde{x}) = x$. By uniqueness of lifts, these two lifts are the same. In particular $\widetilde{y} = \widetilde{f}(1) = h \circ \widetilde{f}(1) = h(\widetilde{y})$. ♠

Combining these two results we see that an element in the kernel of Φ is the identity deck transformation.

6.4 Surjectivity

Let $[g] \in \pi_1(X, x)$ be some element. We want to produce a deck transformation h such that $\Phi(h) = [g]$.

Let $\widetilde{y} \in \widetilde{X}$ be any point. We need to define $h(\widetilde{y})$. So, let $\widetilde{f}$ be a path joining $\widetilde{x}$ to $\widetilde{y}$. Let $f = E \circ \widetilde{f}$. Then f is a path joining x to $y = E(\widetilde{y})$. Consider the concatenated path $\gamma = g * f$. From the lifting property we can find a lifted path $\widetilde{\gamma}$ which joins $\widetilde{x}$ to some other point, which we define as $h(\widetilde{y})$.

Exercise 8: Illustrate the above construction with a series of careful pictures on your favorite example.

Exercise 9: Show that the definition of $h(\widetilde{y})$ is independent of the choices of f and g.

In case $\widetilde{y} = \widetilde{x}$ we can take $\widetilde{f}$ to be the trivial path. In this case $\widetilde{\gamma}$ is a path joining $\widetilde{x}$ to $h(\widetilde{x})$ and $E \circ \widetilde{\gamma}$ differs from $g = E \circ \widetilde{g}$ just by concatenating the constant loop. Assuming that h is a deck transformation, we have $\Phi(h) = [\gamma] = [g]$.

Lemma 6.4 *$E \circ h = E$.*

Proof: Let's compute $E \circ h(\widetilde{y})$. We know that $\widetilde{\gamma}$ connects $\widetilde{x}$ to $h(\widetilde{y})$. Then $\gamma = E \circ \widetilde{\gamma}$ connects x to y. Hence

$$E \circ h(\widetilde{y}) = E \circ \widetilde{\gamma}(1) = E \circ \widetilde{f}(1) = f(1) = y.$$

On the other hand $\widetilde{f}$ is a lift of f. Hence

$$E(\widetilde{y}) = E \circ \widetilde{f}(1) = f(1) = y.$$

This shows that $E \circ h(\widetilde{y}) = E(\widetilde{y})$. Since $\widetilde{y}$ is arbitrary, we are done. ♠

Lemma 6.5 *h is continuous.*

Proof: Let $\widetilde{y} \in \widetilde{X}$ be a point. Let $y = E(\widetilde{y})$. There is an evenly covered neighborhood $U \subset X$ of y. Let $\widetilde{U}_1$ be the component of $h^{-1}(U)$ which contains $\widetilde{y}$. Let $\widetilde{U}_2 = h(\widetilde{U}_1)$. Then $\widetilde{U}_2$ is another component of $h^{-1}(U)$ because $E \circ h = E$. Let F_j be the inverse of the restriction of E to $\widetilde{U}_j$. Then $h = F_2 \circ E$ on $\widetilde{U}_1$. Being the composition of continuous maps, h is continuous. ♠

Were we to make the above construction for the element $[g]^{-1}$ we would produce the map h^{-1}. Hence h is invertible. We know that h is continuous and the same argument shows that h^{-1} is continuous. Hence h is a homeomorphism. This completes our proof.

7 The Last Exercise

Exercise 10: Go through all the examples you know and verify the isomorphism theorem.