Math 20 Midterm 2 Solutions. 19 Nov 2010

1. Switching the order of integration gives

$$\int_{y=0}^{1} \int_{x=0}^{y^2} x \cos(\pi y^5/2)\ dx\ dy = \int_{y=0}^{1} (y^4/2) \cos(\pi y^5/2) dy.$$

Now make the substitution $u = \pi y^5/2$ and $du = (5\pi/2)y^4\ dy$. This leads to the integral

$$\frac{1}{5\pi} \int_{u=0}^{\pi/2} \cos(u)\ du = \frac{1}{5\pi}.$$

2. First let's work out the integrand. The paraboloid can be parametrized as $r(u, v) = (u, v, u^2 + v^2)$. We have

$$\partial r/\partial u = (1, 0, 2u); \qquad \partial r/\partial v = (0, 1, 2v).$$

This leads to

$$\left\| \partial r/\partial u \times \partial r/\partial v \right\| = \sqrt{1 + 4u^2 + 4v^2} = \sqrt{1 + 4r^2}.$$

The last expression is in polar coordinates.

To find the domain of integration, solve the equation $x^2 + y^2 = 2x$. This leads to $(x - 1)^2 + y^2 = 1$, which is a circle of radius 1 centered at the point $(1, 0)$. In terms of polar coordinates, we have $r^2 = 2r \cos(\theta)$, which gives $r = 2 \cos(\theta)$. When θ runs from $-\pi/2$ to $\pi/2$, the region is traced out once. So, the final integral is

$$\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} \sqrt{1 + 4r^2}\ r\ dr\ d\theta.$$

3. a. The centroid is $(1, c)$. When revolved about the x-axis (in space) the centroid moves $2\pi c$. The half-disk has area $\pi/2$. By Pappus's theorem, we have

$$4\pi/3 = 2\pi c \times (\pi/2)$$

This gives $c = 4/(3\pi)$.

b. First put R in a coordinate system to that the vertices are $(0,0)$, $(2,0)$, $(0,-2)$ and $(2,-2)$. Think of R as the union of two pieces, R_1 and R_2, as indicated by the shading in Figure 2.

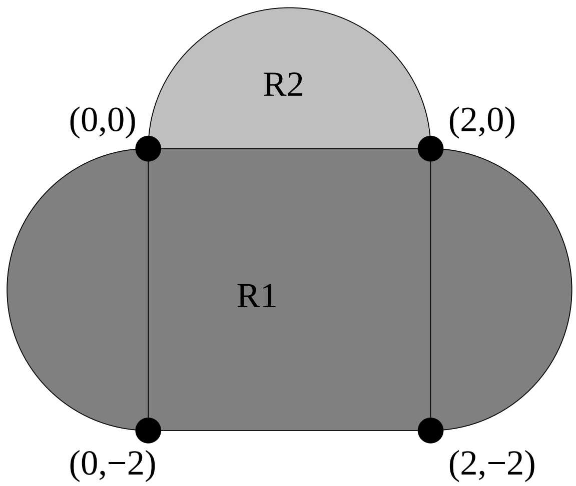

Figure 2: The region R.

The region R_1 has centroid $(1,-1)$, by symmetry, and mass $4+\pi$. The region R_2 has mass 2π, and centroid $(1,c)$, where $c = 4/(3\pi)$. The first coordinate of the centroid is 1. The second coordinate is

$$\frac{(-1)(4+\pi)+(4/(3\pi))(\pi/2)}{4+\pi+\pi/2} = \frac{-10/3-\pi}{4+3\pi/2}.$$

4. Let S be the region described in the problem. Call the transformation T. The region $R = T(S)$ is the rectangle bounded by the lines $x = 1, 2$ and $y = 4, 8$. The Jacobian is

$$J_T = \det \begin{bmatrix} 2u & -2v \\ v & u \end{bmatrix} = 2(u^2+v^2).$$

So

$$\int_S (u^2+v^2)dudv = \frac{1}{2}\int_S J_T = \frac{1}{2}\int_R 1dxdy = \frac{1}{2}\times 1\times 4 = 2.$$