Math 2410 HW1

Due Tuesday, Oct 5.

1: This is a theoretical result and an application. (a) Let M be metric space. Suppose that every nested sequence of closed subsets of M, namely $A_1 \supset A_2 \supset A_3 ...$ has nonempty intersection: $\bigcap A_n \neq \emptyset$. Suppose also that for any $\epsilon > 0$ there is some constant n_ϵ such that a ball of radius ϵ is covered by at most n_ϵ balls of radius $\epsilon/2$. Prove that a closed and bounded subset of M is compact.

(b) Let M denote the set of closed subsets of the unit square in the plane. Given two elements of M, namely $C_1, C_2 \subset [0,1]^2$, define $d(C_1, C_2)$ to be the infimal (i.e. smallest) $\epsilon > 0$ such that every point of C_1 is within ϵ of C_2 and *vice versa*. Prove that M is a compact metric space. This metric space is sometimes called *hyperspace*.

2: Prove the following results:

- The sphere and the surface of a cube are homeomorphic.
- The cylinder and the Moebius band are homotopy equivalent.
- The cylinder and the Moebius band are nor homeomorphic.
- The figure 8 is homotopy equivalent to $\boldsymbol{R}^3 - R_1 - R_2 - R_3$ where each R_j is a ray starting at the origin, and the three rays are distinct.
- The Lie group $SO(3)$ (or orientation preserving orthogonal matrices) is homeomorphic to $\boldsymbol{RP}^3$.

3: This is essentially problem 1.1.9 in Hatcher. Let A_1, A_2, A_3 be three compact subsets of $\boldsymbol{R}^3$. Use the Borsuk-Ulam Theorem to prove that there exists a plane which divides all 3 regions into sets of equal volume. (If you don't like to think about the volume of an arbitrary compact set, you can think of each A_i is a finite union of cubes.)

4: These examples are called *Lens spaces*. Let $\omega = \exp(2\pi i/n)$, where $n \geq 3$ is some integer. Here ω is an n-th root of unity. Let S^3 denote the unit sphere in $\boldsymbol{C}^2$. Define an equivalence relation on S^3 by the rule that

$(z, y) \sim (\omega^k z, \omega^k w)$ for each choice $n = 0, ..., n-1$. Let $L(n) = S^3 / \sim$ be the quotient space. Prove that $\pi_1(L(n))$ is isomorphic to $\boldsymbol{Z}/n$.

5: Let A, B, C all be copies of the projective plane. Let a, b, c respectively be points in A, B, C. Let X be the space obtained by gluing a, b, d together. So, X is a "wedge" of 3 projective planes. Prove that $\pi_1(X)$ contains the free group on 2 generators as a subgroup.

6: This is problem 2.1.17 in Hatcher. Show that $\pi_1(\boldsymbol{R}^2 - Q^2)$ is uncountable.