The Isomorphism Theorem

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The purpose of these notes is to sketch the theorem that I will prove in class on Tuesday. One reason I want to give you these notes now is because it may help with the homework. The other reason is that these notes are sort of a high-level description of the main ideas in the big theorem. If you read them in advance of class, the lecture will probably make more sense.

1 The Universal Cover

Let X be a topological space. Let $\widetilde{X}$ be a covering space. This means that there is a surjective continuous map $\phi : \widetilde{X} \to X$ such that each point in x has an open neighborhood U such that $\phi^{-1}(U)$ consists of a disjoint union of sets $\{U_\alpha\}$ where $\phi : U_\alpha \to U$ is a homeomorphism for each U_α.

The space $\widetilde{X}$ is called a *universal covering space* if $\pi_1(X)$ is the trivial group. This is often expressed by saying that $\widetilde{X}$ is *simply connected*. Most of the examples I have shown in class are universal covers. When $\widetilde{X}$ is a covering space, the *deck group* is the group of homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $\phi \circ h = \phi$. In other words, these homeomorphisms commute with the covering map.

Why is this called the deck group. I think that this name comes from the idea that $\widetilde{X}$ is a deck of cards and X is just a single card. The map ϕ in this case is just projection. The deck group is just the group that effects shuffles of the cards. I don't really like this example because $\widetilde{X}$ is pretty boring: It is a disconnected space. The connected covering spaces are more fun for our purposes.

Here are some examples.

1. $\widetilde{X} = \boldsymbol{R}$ and $X = S^1$ and $\phi(x) = \exp(ix)$. In this case the deck group is $\boldsymbol{Z}$. The homeos all have the form $h(x) = x + n$ for $n \in \boldsymbol{Z}$. So, the deck group is isomorphic to $\boldsymbol{Z}$. Coincidentally, $\pi_1(S^1) = \boldsymbol{Z}$ as well.

2. $\widetilde{X}$ is the infinite 4-valent tree and X is the figure-8, The map ϕ a bit hard to explain with a formula, but basically it is a map that maps labelled edges in $\widetilde{X}$ to one or the other loop in X, depending on the label. The deck group is the group of all label-preserving automorphisms of the tree. This is the free group on 2 generators. Coincidentally, $\pi_1(X)$ is also the gree group on 2 generators.

3. $\widetilde{X} = S^2$ and X is the projective plane, $\boldsymbol{P}^2$. The map ϕ is the quotient by the antipodal map. The deck group is $\boldsymbol{Z}/2$. The generator is the antipodal map $A(x) = -x$. The other element is the identity map. Coincidentally, $\pi_1(\boldsymbol{P}^2) = \boldsymbol{Z}/2$ as well.

1.1 The Main Result

You might have noticed a coincidence going on: $\pi_1(X)$ is isomorphic to the deck group in these examples. This turns out to always be the case.

Theorem 1.1 *Suppose that $\widetilde{X}$ is a universal covering space of a path connected space X. Then the deck group of $(\widetilde{X}, X, p)$ isomorphic to the fundamental group of X.*

The proof is essentially the same as what we did in class for S^1. Here is a sketch of the proof. We choose a basepoint $p \in X$ and also a point $\widetilde{p} \in \widetilde{X}$ such that $\phi(\widetilde{p}) = p$. We represent each element of $\pi_1(X, p)$ by a map $\gamma : [0, 1] \to X$ such that $\gamma(0) = \gamma(1) = p$. We then give essentially the same argument as in class to show that there exists a unique lift $\widetilde{\gamma}$ of γ such that that $\widetilde{\gamma}(0) = \widetilde{p}$. Our homomorphism sents $[\gamma]$ to the deck transformation that maps $\widetilde{p} = \widetilde{\gamma}(0)$ to $\widetilde{\gamma}(1)$. That's the construction. Call the map Ψ. Here are the main details:

1. Why is there is always a deck transformation that takes $\widetilde{p}$ to $\widetilde{\gamma}(1)$? In other words, why does Ψ even have a target to map into? The proof that this works is going to be a variant of the path lifting property. We can *start* building a homeo in the following way. Near $\widetilde{p}$ we compose ϕ with the correct branch of ϕ^{-1}. This defines our homeo in a neighborhood of

$\widetilde{p}$. We then use a kind of "continuation" property related to the path lifting to define the homeo everywhere.

2. Why is Ψ well defined? This is going to work about the same way as it did for the circle: Homotopies lift as well.

3. Why is Ψ a homomorphism? This is because concatenation interacts the right way with the deck group. Suppose that we have a path $\gamma_1 * \gamma_2$. Then the lifted concatenation is the concatenation of the lifts: The lift $\widetilde{\gamma}_1$ starts at $\widetilde{p}$ and ends at $\Psi(\gamma_1)(\widetilde{p})$. The continuation of the lift is the image of $\widetilde{\gamma}_2$ under the map $\Psi(\gamma_1)$. So, it starts at $\Psi(\gamma_1)(\widetilde{p})$ and ands at $\Psi(\gamma_1)(\Psi(\gamma_2(\widetilde{p}))$. The whole concatenation connects $\widetilde{p}$ to $\Psi(\gamma_1) \circ \Psi(\gamma_2)(\widetilde{p})$. It turns out that an element of the deck group is determined by where it sends $\widetilde{p}$. Hence $\Psi(\gamma_1\gamma_2) = \Psi(\gamma_1)\Psi(\gamma_2)$.

4. Why is Ψ surjective? Because you can draw a path from $\widetilde{p}$ to any other point in the orbit of the deck group and then project to X. This gives you the loop you want to lift to get to the point of interest to you.

5. Why is Ψ injective? Well, if $\Psi(\gamma)$ is the identity map then $\widetilde{\gamma}$ is a closed loop in $\widetilde{X}$ that starts and stops at $\widetilde{p}$. But since $\pi_1(\widetilde{X})$ is trivial, there is a homotopy $\widetilde{H}$ from $\widetilde{\gamma}$ to the trivial loop. We push down the homotopy and use it to show that γ is the trivial element as well.

The main point of Tuesday's lecture will be to prove this theorem and work out the above details.