Math 2410 HW1

Due Thursday, Oct 6.

1: The definition of Antoine's necklace on (say) wikipedia is a bit vague. This exercise gives a concrete construction. Say that a *round torus* is a subset of $\boldsymbol{R}^3$ consisting of all points which are within some ϵ of a round circle C. Here ϵ needs to be less than half the radius of C. The curve C is called the *core* of the round torus. Prove that there exists a round torus T, an integer K, and a finite list $T_1', ..., T_K' \subset T$ of round tori with the following properties:

- Each T_i' and T_j' are disjoint and isometric to each other.
- For all i there is a distance decreasing similarity f_i such that $T_i' = f_i(T)$. other words T_i' and T have the same shape.
- T_i' and T_{i+1}' are linked for all i, with indices taken cyclically. What this means is the the core circle of T_i' intersects once the disk spanning the core circle of T_{i+1}' and vice versa.
- T_i' and T_j' otherwise are unlinked.
- The set of centers (of mass) of $T_1', ..., T_K'$ make a regular K-gon that lies in the same plane as the core of T and surrounds it. All this is to say that $T_1' \cup ... \cup T_K'$ links T in the pattern that you would see if you looked up Antoine's necklace on wikipedia. (Please do!)

It is probably easiest to take K large. I'm not sure about the smallest K which would work.

2: Let T and $T_1', ..., T_K'$ be as in Exercise 1. Let $T_0 = T$. Assuming that T_n has been defined, let T_{n+1} be obtained as follows: For each round torus $A \subset T_n$ let $f : T \to A$ be a similarity so that $A = f(T)$. Replace A by $A' = f(T_1' \cup ... T_K')$. You are supposed to do this for all such A. Now that all these sets have been defined inductively, The intersection $T_\infty = \bigcap T_n$ is called *Antoine's necklace*. Prove that T_∞ is homeomorphic to the middle-third Cantor set. Hint: It would probably be easiest to first prove that T_∞ is homeomorphic to the Cantor set you get by iteratively cutting an interval into K (rather than 3) pieces, and then showing that this Cantor set is homeo. to the middle-third Cantor set.

3: Prove the following results:

- Any two compact convex subsets of $\boldsymbol{R}^3$, with non-empty interiors, are homeomorphic. (Examples are closed balls and solid cubes.)
- The cylinder and the Moebius band are homotopy equivalent but not homeomorphic.
- The figure 8 is homotopy equivalent to $\boldsymbol{R}^3 - R_1 - R_2 - R_3$ where each R_j is a ray starting at the origin, and the three rays are distinct.
- The Lie group $SO(3)$ (or orientation preserving orthogonal matrices) is homeomorphic to $\boldsymbol{RP}^3$.

4: This is essentially problem 1.1.9 in Hatcher, but specialised a bit to make it easier. Say that a *blob* is a finite union of solid closed cubes which homeomorphic to a cube. Suppose that A_1, A_2, A_3 is a collection of 3 blobs such that $A_1 \cup A_2 \cup A_3$ is also a blob. Use the Borsuk-Ulam Theorem to prove that there exists a plane which divides all 3 individual blobs (meaning A_1, A_2, A_3) into sets of equal volume. In fact this result works for any sets A_1, A_2, A_3 but taking everything in sight to be a blob means that we don't have to fool around with measure theory to understand what *volume* means. You can deduce the general case from the blob case by taking a sequence of blob approximations which converges in measure to the general sets.

4: These examples are called *Lens spaces*. Let $\omega = \exp(2\pi i/n)$, where $n \geq 3$ is some integer. Here ω is an n-th root of unity. Let S^3 denote the unit sphere in $\boldsymbol{C}^2$. Define an equivalence relation on S^3 by the rule that $(z, y) \sim (\omega^k z, \omega^k w)$ for each choice $n = 0, ..., n-1$. Let $L(n) = S^3/\sim$ be the quotient space. Prove that $\pi_1(L(n))$ is isomorphic to $\boldsymbol{Z}/n$.

5: Let A, B, C all be copies of the projective plane. Let a, b, c respectively be points in A, B, C. Let X be the space obtained by gluing a, b, d together. So, X is a "wedge" of 3 projective planes. Prove that $\pi_1(X)$ contains the free group on 2 generators as a subgroup.

6: This is problem 2.1.17 in Hatcher. Show that $\pi_1(\boldsymbol{R}^2 - Q^2)$ is uncountable.