# Math 2410 HW2

Due Thursday, Nov 3.

**1a:** Prove that Antoine's necklace $A$ is homeomorphic to the middle third Cantor set without using any results about the classification of perfect, totally disconnected metric spaces.

**1b: (Extra credit)** Assuming any results you like in knot theory concerning the linking of loops in $\boldsymbol{R}^3$, prove that $\boldsymbol{R}^3 - A$ is not simply connected. This is the amazing property of Antoine's necklace: It is homeo to a Cantor set but it's complement in space is not simply connected.

**2:** Without using any results about the classification of subgroups of free groups, prove that any finite index subgroup of a finitely generated free group is a free group. (Hint: Pick a good space and then use the Galois correspondence.)

**3:** Prove that the universal cover of a genus 2 surface can be realized as an increasing union $U = U_1 \cup U_2 \cup U_3 ...$ where

- $U_k$ is simply connected for all $k$.
- $U_1$ is a regular octagon.
- $U_{k+1}$ is obtained by gluing a regular octagon to $U_k$ along some portions of the boundaries of these sets, for all $k$.

You can (but don't need to for the problem) use these properties to prove that $U$ is homeomorphic to an open disk.

**4:** Put a $\Delta$-complex structure on the Klein bottle and compute its simplicial homology. (This is basically Problem 2.1.5 in Hatcher's book.)

**5:** Suppose that $X$ is a smooth compact manifold with a metric that makes it locally isometric to 3-dimensional Euclidean space. Prove that $X$ is homeomorphic to a simplicial complex. (Note that topologically $X$ need not be a 3-torus; there are some other topological types as well.) Hint: look up Voronoi cells on wikipedia.

**6:** Without using any general results about triangulations of smooth compact manifolds, and without using any known explicit triangulation of $\boldsymbol{CP}^2$, show that the complex projective plane $\boldsymbol{CP}^2$ has a triangulation – i.e. is homeomorphic to a simplicial complex. Your triangulation doesn't have to be explicit; you just have to show that it works. Hint: Here is one approach: In homogeneous coords let $B_j$ be the set of points $[z_1 : z_2 : z_3]$ where $|z_j| = \max(|z_1|, |z_2|, |z_3|)$. The sets $B_1 \cup B_2 \cup B_3$ partition $\boldsymbol{CP}^2$...