# Fundamental Groups of 4-Manifolds

Rich Schwartz

September 23, 2021

The purpose of these notes is to sketch the proof of the following result.

**Theorem 0.1** *Every finitely presented group is the fundamental group of a smooth compact* 4*-manifold.*

My proof will skimp on some of the fine points, but I hope that the main ideas are made clear. Also, in defense of this sketchy approach, let me say this: Some of the general technical results, like Lemma 4.1 and 5.1, are hard to prove *in full generality*, but for the concrete constructions needed for the main result, you can do things in such a controlled way that you can just see that the technical points work out. Of course, these notes will be easier for you to read if you already know about smooth manifolds.

## 1 Manifolds

A *compact topological $n$-manifold* is a compact Hausdorff space having the property that every point has an open neighborhood that is homeomorphic to $\boldsymbol{R}^n$. Here are some classic examples:

- The sphere $S^n$ is a compact topological $n$-manifold.
- The torus $\boldsymbol{T}^n = \boldsymbol{R}^n/\boldsymbol{Z}^n$ is a compact topological $n$-manifold.
- The product of a compact topological $n$-manifold and a compact topological $n$-manifold is a compact topological $(n+m)$-manifold.

Suppose we have a compact topological manifold. Our various coordinate charts, giving homeomorphisms from neighborhoods of $M$ into $\boldsymbol{R}^n$, sometimes overlap. If $U$ and $V$ are open sets in $M$ such that $U \cap V$ is non-empty, then the two homeomorphisms $h_U : U \to \boldsymbol{R}^n$ and $h_V : V \to \boldsymbol{R}^n$ define a new function

$$h_U h_V^{-1} : h_V(U \cap V) \to h_U(U \cap V).$$

The manifold is called *smooth* if these maps are always smooth. That is, they are infinitely differentiable. All the examples mentioned above can be made smooth easily.

It is better to work with smooth manifolds than with topological manifolds because it easier to control the process of taking neighborhoods of subsets. If $S \subset M$ is some subset we say that an open neighborhood $N$ of $S$ is *thin* if $N$ is homotopy equivalent to $S$ via a homotopy which is the identity on $S$. Technically this is expressed as saying that $N$ *deformation retracts* to $S$. When this happens $N$ and $S$ have the same fundamental group. If $S$ is some reasonable set, like a smooth sub-manifold, it is easy to construct a thin neighborhood.

# 2 Manifolds with Boundary

If is better to give examples of manifolds with boundary before giving the formal definition, which is a bit complicated. A solid ball is the prototypical example. The sphere is the boundary and the open ball is the interior part. Cylinders and Moebius bands also give examples.

Here is the formal definition. A *compact topological n-manifold with boundary* is a compact Hausdorff space $M$ which can be written as the disjoint union $M^o \cup \partial M$ with the following properties:

1. Every point in $M^o$ has an open neighborhood homeomorphic to $\boldsymbol{R}^n$.

2. Every point in $\partial M$ has an open neighborhood $U$ and a homeomorphism $h : U \to h(U) \subset \boldsymbol{R}^n$ such that $h(U \cap M^o)$ is an open half-space and $h(U \cap \partial M)$ is the hyperplane bounding this half space.

All the examples given above fit into this description.

A *smooth manifold with boundary* is defined in the same way that a smooth manifold is defined. The overlap functions should all be smooth.

Since the domain for these maps are subsets of half-spaces, the differentiability is taken in a 1-sided sense for points on the boundary. (This is the kind of fine point you shouldn't worry about.)

# 3 The Gluing Construction

Here is the neat property of manifolds with boundary. Suppose that we have 2 topological manifolds $M$ and $N$ of the same dimension. Suppose also that we have a homeomorphism $h$ from $\partial M$ to $\partial N$. Then we may form the quotient space

$$S = (M \coprod N)/h$$

obtained by attaching $M$ to $N$ using $h$. (The symbol in the equation means "disjoint union".) The space $S$ is essentially obtained from $M$ and $N$ by gluing them together along their boundaries.

**Lemma 3.1** *The space $S$ is a compact topological manifold.*

**Proof:** A point in $M^o$ still has a neighborhood homeomorphic to $\boldsymbol{R}^n$. The same goes for a point in $N^o$. A point on the "seam" $S - M^o - N^o$ has "half-neighborhoods" on each side (one in $M$ and one in $N$) which fit together to make a neighborhood homeomorphic to $\boldsymbol{R}^n$. Thus, every point of $S$ has an open neighborhood homeomorphic to $\boldsymbol{R}^n$. ♠

The same constructions and result work in the case of smooth manifolds. The main difference is that *homeomorphism* is replaced by *diffeomorphism*. When we do things smoothly, the lemma above implies that we get a topological manifold. The fact that the overlap functions are smooth is a bit tricky but in the applications we have in mind the spaces are so concrete that it is easy to arrange.

This gluing construction is the basis for various operations that generally go under the heading of *surgery*. I won't define what this means in general, but I will give the two examples needed for the proof of the main result.

# 4 Connect Sum

The first operation is called *connect sum*. Given 2 compact manifolds $M$ and $N$, we first create manifolds-with-boundary $M'$ and $N'$ by deleting small open balls from $M$ and $N$. The boundaries of $M'$ and $N'$ are both spheres and so there is a homeomorphism mapping the one to the other. When we do this the resulting space is called the *connect sum* of $M$ and $N$ and it is written as $M\#N$.

The construction just made involves many choices. We have to choose which balls to cut out and then we have to choose a homeomorphism which glues the newly created boundaries together. It turns out that all choices lead to the same manifold, up to homeomorphism. You can play around with this construction. For instance, try to convince yourself that the connect sum of two tori is a surface of genus 2 no matter how you do it. You can also do the construction for smooth manifolds, using diffeomorphisms rather than homeomorphisms. Again, the spaces and operations needed for the main result are so concrete that the fine points work out easily.

Fortunately for us, the main result does not depend on the fact that we always get the same space when we do connect sum. Assuming we start with connected manifolds, we can state our next result without mentioning a basepoint because we only care about the groups up to isomorphism.

**Lemma 4.1** *Let $n \geq 3$. If $M$ and $N$ are compact smooth $n$-manifolds of dimension at least 3, then $\pi_1(M\#N) = \pi_1(M) * \pi_1(N)$.*

**Proof:** Note first of all that $M'$ is homotopy equivalent to $M$ with one point deleted. When $M$ has dimension at least 3 deleting this point has no effect on the fundamental group. That is, $\pi_1(M') = \pi_1(M)$. Likewise $\pi_1(N') = \pi_1(N)$.

Now we set up an application of Van Kampen. To get our space $A$ we start with $M'$, considered as a subset of $M\#N$, and we take a thin neighborhood so that the result is open in $M\#N$. So, basically $A$ is just $M'$ but with a little bit of extra padding to accommodate the openness condition in Van Kampen's Theorem. Likewise the space $B$ is a thin neighborhood of $N'$. If we do this in a sensible way then $A \cap B$ is a thin neighborhood of a sphere.

We choose our basepoint $p \in A \cap B$. The group $\pi_1(A \cap B, p)$ is trivial because $\pi_1(S^n)$ is trivial for $n \geq 2$. Van Kampen's Theorem now tells us that $\pi_1(A \cup B, p) = \pi_1(A, p) * \pi_1(B, p)$. But $A \cup B = M\#N$. Hence

$$\pi_1(M\#N) = \pi_1(A, p) * \pi_1(B, p) = \pi_1(M') * \pi_1(N') = \pi_1(N) * \pi_1(M).$$

This does it. ♠

## 5 Dehn Surgery

Let $B^n$ denote the ball of dimension $n$. The product $B^{n+1} \times S^m$ is a manifold with boundary. The boundary is $S^n \times S^m$, the product of 2 spheres. Here is the neat fact: Both spaces $B^{n+1} \times S^m$ and $S^n \times B^{m+1}$ have the same boundary up to homeomorphism, namely $S^n \times S^m$.

Now let us tune these numbers. The two spaces $B^3 \times S^1$ and $B^2 \times S^2$ have the same boundary. The first space has fundamental group $\boldsymbol{Z}$ and the second one has trivial fundamental group. This fact gives us a way to modify the fundamental group of a compact 4-manifold.

The input is a smooth 4-manifold $M$ and a smooth embedded loop $L \subset M$ representing some element of the fundamental group. We choose a thin neighborhood $\Lambda$ of $L$ which is homeomorphic to $S^1 \times S^3$. Now we cut out $\Lambda$. This gives a manifold whose boundary is $S^1 \times S^2$. Next, we glue in a copy of $D^2 \times S^2$ along the boundary. We can do this because the space we have cut out and the space we sew back in have the same boundary up to diffeomorphism. Let $\widehat{M}$ be the new manifold.

**Lemma 5.1** *Let $N_L$ be the normal closure of the element of $\pi_1(M)$ represented by $L$. Then $\pi_1(\widehat{M}) = \pi_1(M)/N_L$.*

**Proof:** Two loops in a 4-dimensional manifold do not link each other, much in the same way that a loop in 3-dimensional space does not link a point. So any loop in $\pi_1(M)$ is homotopic to one in $\pi_1(M - \Lambda)$ and any homotopy in $M$ between loops in $M - \Lambda$ can be modified so that it misses $\Lambda$. For this reason, the inclusion of $\pi_1(M - \Lambda)$ into $\pi_1(M)$ is an isomorphism. In short, $\pi_1(M) = \pi_1(M - \Lambda)$.

We let $A$ be a thin neighborhood of $M - \Lambda$ in $\widehat{M}$. Let $B$ be a thin neighborhood of $D^2 \times S^2$ in $\widehat{M}$. Then $A \cap B$ is a thin neighborhood of $S^1 \times S^2$. This means that $\pi_1(A \cap B) = \boldsymbol{Z}$. At the same time $\pi_1(A) = \pi_1(M)$ and $\pi_1(B)$ is trivial. The inclusion of $\boldsymbol{Z} = \pi_1(A \cap B)$ into $\pi_1(A)$ maps the generator, 1, to the element represented by $L$. Therefore, this inclusion maps $\boldsymbol{Z}$ to powers of the element in $\pi_1(A)$ represented by $L$. By Van Kampen,

$$\pi_1(\widehat{M}) = \pi_1(A \cup B) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/N = \pi_1(M)/N.$$

Here $N$ is the normal closure of the set of elements of the form $gh^{-1}$ where $g \in \pi_1(A)$ is a power of the element represented by $L$ and $h \in \pi_1(B)$ is just trivial. In other words $gh^{-1} = g$ and so $N = N_L$. ♠

# 6 Putting it Together

Suppose $G$ is some finitely presented group. This means that $G$ has a description of a gree group on $k$ generators modulo the normal closure of some list of $\ell$ words, the relators. That is

$$G = \langle a_1, ..., a_k | r_1, ..., r_\ell \rangle$$

We start with $S^4$ and then connect sum it with $S^1 \times S^3$ a total of $k$ times. So, we have all these copies of $S^1 \times S^3$ attached to a central hub. Call the resulting 4-manifold $M$. Repeated applications of Lemma 4.1 tell us that $\pi_1(M)$ is the free group on $k$ generators. That is:

$$\pi_1(M) = \langle a_1, ..., a_k | \ \rangle$$

Now, we represent each relator $r_j$ by some smooth embedded loop $L_j$ in $M$. We can make all these loops disjoint from each other. For each loop $L_j$ we perform the Dehn surgery as described above. The first time we do it, we get a manifold $M'$ such that

$$\pi_1(M') = \pi_1(M)/N_{L_1} = \langle a_1, ..., a_k | r_1 \rangle$$

by Lemma 5.1. The second time we do it, we get a manifold $M''$ such that

$$\pi_1(M'') = \pi_1(M')/N_{L_2} = \langle a_1, ..., a_k | r_1, r_2 \rangle.$$

Continuing in this way we finally arrive at a smooth compact 4-manifold whose fundamental group is exactly $G$.