A Homotopy Proof

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The purpose of these notes is to work out a simple homotopy proof in a lot of detail. I will prove that the punctured plane is homotopic to the unit circle. I don't expect this much detail on the homework, but on the homework you should break the proof down into reasonably small steps which seem pretty clear and doable.

Definition of the Maps: Translations of the plane give homeomorphisms between a plane punctured at the origin and a plane punctured at another point. So, without loss of generality, we can assume that the plane is punctured at the origin. Let $X = \boldsymbol{R}^2 - \{(0,0)\}$ and let Y be the unit circle in X. Let $f : X \to Y$ be the map given by

$$f(p) = p/\|p\|$$

and let $g : Y \to X$ be the inclusion map. The map f is well-defined because $\|p\| > 0$ for all $p \in X$.

Continuity of the Maps: The inclusion map g is continuous by definition: If $U \subset X$ is open then $g^{-1}(U) = U \cap Y$, a set which is open by definition of the subspace topology. To show that f is continuous it suffices to check this on a basis for the topology of the circle Y, namely on open intervals. If I is an open interval of Y then $f^{-1}(I)$ is an open cone bounded by the rays emanating from the origin and containing the endpoints of I. Open cones are pretty clearly open in X.

Definition of the Homotopies: It remains to show that the compositions $f \circ g$ and $g \circ f$ are homotopic to the identity on Y and X respectively. The composition $f \circ g$ is the identity on Y. There is nothing more to say.

Now we consider $g \circ f$. This is the hardest part of the proof. Intuitively we push the plane a little bit towards the circle at a time. The domain for the homotopy is $X \times [0,1]$, which we think of as a subset of $\boldsymbol{R}^3$.

Define

$$\|p\|_t = (1-t) + t\|p\|.$$

By construction $\|p\|_0 = 1$ and $\|p\|_1 = \|p\|$. Also, $\|p\|_t > 0$ for all $p \in X$ and all $t \in [0,1]$ because $\|p\| > 0$. Finally, define

$$H(p,t) = p/\|p\|_t.$$

Homotopy Continuity: I will give the proof in 4 steps. The high level idea is to separately study the effect of fixing the time and varying the point and then the effect of fixing the point an varying time. Once we have this information we use the triangle inequality to put it together. We first choose a big compact set to use, so that we can make use of compactness results.

Step 1: Let $(p,t) \in X \times [0,1]$ be a point. Choose a compact subset

$$K \subset X \times [0,1]$$

containing (p,t). There is some $d > 0$ so that all $p \in K$ satisfies $\|p\| \in [d, 1/d]$.

Step 2: First we study the effect of fixing the time and varying the point. Since $\|p\|_t \in [1, \|p\|]$, the map $p \to p/\|p\|_t$ expands distances by at most a factor of $C_1 = 1/d$. In particular, if $p \in K$ then our map satisfies

$$\|H(p,t) - H(p^*,t)\| < C_1\|p - p^*\|. \tag{1}$$

for all $p, p^* \in K$ and all $t \in [0,1]$.

Step 3: Now we study the effect of fixing the point and varying the time. For each $p \in K$ the map

$$h_p(t) = p/\|p\|_t$$

is differentiable because it is the composition of differentiable maps

$$h_1(p,r) = rp, \qquad h_2(s) = 1/s, \qquad h_3(t) = (1-t) + \|p\|.$$

The size of the derivative of the composition only depends on $\|p\|$, essentially by the Chain Rule. Since $\|p\| \in [d, 1/d]$ we have some constant C_2 such that $\|h'_p(t)\| < C$ for all $p \in K$ and all $t \in [0,1]$. Integrating, we see that

$$\|h_p(t) - h_p(t^*)\| \leq C_2|t - t^*| \tag{2}$$

for all $p \in K$ and $t, t^* \in [0,1]$.

Step 4: We use the triangle inequality to assemble Steps 2 and 3. The last step of the proof is the triangle inequality. Combining Equations 1 and 2 and using the triangle inequality:

$$\|H(p^*,t^*) - H(p,t)\| \leq \|H(p^*,t^*) - H(p,t^*)\| + \|h_p(t) - \|h_p(t^*)\| \leq$$

$$C_1\|p_* - p\| + C_2|t - t^*| \leq$$
$$(C_1 + C_2)\|(p,t) - (p^*,t^*)\|.$$

This shows that H is continuous: It only expands distances by at most a factor of $C_1 + C_2$ in a neighborhood of (p,t).