Notes on π_1 and H_1

Rich Schwartz

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The purpose of these notes is to explain the "tadpole" construction better, and then to sketch the proof that H_1 is the abelianization of π_1. This proof is written out more fully and carefully in Hatcher's book, but I hope that the notes make the main ideas more clear.

1 The Tadpole Result

Here is the input for the construction:

- X is a path connected space and $p \in X$.
- Σ is a compact surface with boundary.
- $K \subset \Sigma$ is some proper compact subset.
- $p^* \in \Sigma - k$ is some point.
- $f : \Sigma \to X$ is a continuous map.

The output is a new continuous map $g : \Sigma \to X$ such that $g = f$ on K and $g(p^*) = p$. In the application from class, K is the boundary of Σ and p^* is the basepoint for $\pi_1(\Sigma, p^*)$ and p is the basepoint for $\pi_1(X, p)$.

Step 1: Choose small closed disks $\Delta \subset \Delta'$, disjoint from K and centered at p^*. Now choose a homeomorphism h from $\Sigma - \Delta$ to $\Sigma - p^*$ which is the identity outside Δ'. The basic idea is that you first make a homeomorphism from the annulus $\Delta' - \Delta$ to the punctured disk $\Delta' - p^*$ by widening all the radial segments emanating from p^*. Then you extend the map to be the identity outside of Δ'.

Step 2: Consider the new map $f_2 : \Sigma - \Delta \to X$ defined by $f_2 = f \circ h$. Here f_2 maps all of the boundary $\partial\Delta$ to a single point, say $q \in X$. Now extend f_2 so that it maps all of Δ to q. The final map f_2 agrees with f on K and maps all of Δ to q.

Step 3: Let $\gamma : [0,1] \to X$ be a path which connects q to p with $\gamma(0) = p$ and $\gamma(1) = q$. Let $\Delta(t)$ be the family of concentric circles in Δ interpolating between q^* and $\partial\Delta$. Here $\Delta(0)$ (by a slight abuse of terminology) is just p^* and $\Delta(t) = \partial\Delta$. Define g so that $g = f^2$ outside Δ and $f(\Delta(t)) = \gamma(t)$. By construction g is continous and $g(p^*) = p$ and $g = f$ on K. That's the end of the construction.

2 Abelianization of the Fundamental Group

Here is a sketch of the proof that $H_1(X)$ is the abelianization of $\pi_1(X,p)$ when X is a path connected space.

Step 1: Construct a map $\Psi : \pi_1(P,p) \to H_1(p)$. Given any $[f] \in \pi_1(P,p)$ we just treat f as a singular chain. After all, f is a map from the simplex $[0,1]$ to X.

Step 2: Show that $\Psi([f])$ is independent of the representative. If f and g are homotopic, then you can divide the square into 2 triangles A and B and consider the chain $F|_A + F|_B$. The boundary of this chain is $f - g$.

Step 3: Say that a *bigon* is the image of a triangle which maps one edge to some path γ and the other two edges respectively to paths γ_1 and γ_2 whose concatenation is the reverse of γ. Bigons are boundaries. If we compare $\Psi(fg)$ with $\Psi(f) + \Psi(g)$ we see that it is really just a bigon: We do f, to g, then return along $(fg)^{-1}$. This shows that $\Psi(fg)$ and $\Psi(f) + \Psi(g)$ differ by a boundary. Hence Ψ is a homomorphism.

Step 4: If $\alpha \in Z_1$ is some cycle, then we first write α as a finite sum of *elementary cycles*. An elementary cycle is one where all the coefficients are 1. Geometrically an elementary cycle is just a loop with all the individual edges "labelled" by 1s. Since the image of Ψ is a group, it suffices to prove that an elementary cycle lies in the image of Ψ. So, suppose α is an elementary cycle. We modify α by adding bigons connecting some vertex v of α to the base point p. We then interpret the resulting *lasso*. As a path, it is $\gamma\alpha\gamma^{-1}$ where γ is a path connecting p to v. as the image of some element of the fundamental group. Hence Ψ is surjective.

One fine point of this construction is that our lasso path is an elementary cycle involving perhaps many individual edges. Iteratively using the same trick as in Step 2, you replace the lasso by one that is made from a single edge. The new lasso is literally in the image of

Step 5: Since $H_1(X)$ is abelian, we see that the commutator of $\pi_1(X, p)$ lies in the kernel.

Step 6: Pick some element $\alpha \in \pi_1(X, p)$ that is in the kernel of Ψ. We want to see that α lies in the commutator subgroup.

There is some 2-chain $\sum n_i \sigma_i$ whose boundary is α. If we allow the same simplex to be listed multiple times we can assume all the n_i are ± 1. Given $f_j : \Delta_j \to X$ for $j = 1, 2$ and edges $\sigma_j \in \Delta_j$ we glue σ_1 to σ_2 if and only if $f_1|_{\sigma_1} = f_2|_{\sigma_2}$. Given that the boundary of our chain is just α, the result of all our gluings is a surface Σ with boundary and a map $f : \Sigma \to X$ such that $f(\partial\Sigma) = \gamma$. That is the main idea of the proof.

Choose a basepoint $p^* \in \Sigma - \partial\Sigma$. Using the tadpole construction we can assume that $f(p^*) = p$. We also know that f maps at least one point $v \in \partial\Sigma$ to p because, after all, $f(\partial\Sigma) = \alpha$ is an element of $\pi_1(X, p)$. Choose a path $\beta \in \Sigma$ which joins p^* to v. By construction, $\beta(\partial\Sigma)\beta^{-1}$ is a loop on Σ, based at p^*, and $f(\beta) \in \pi_1(X, p)$.

Since Σ is an oriented surface with boundary, we can represent Σ with a standard gluing diagram – a polygon whose edge labels are

$$a, b, a^{-1}b^{-1}c, d, c^{-1}d^{-1}, ...$$

The loop $\beta(\partial\Sigma)\beta^*$ is homotopic to $[a, b][c, d]...$ the product of commutators, *via* a homotopy which preserves p^*. But then

$$f(\beta(\partial\Sigma)\beta^{-1}) = f(\beta)\alpha f(\beta)^{-1} = [f(a), f(b)][f(c), f(d)]...$$

Hence α is conjugate to a product of commutators. This places α in the commutator subgroup.