Conway's Nightmare: Brahmagupta and Butterflies

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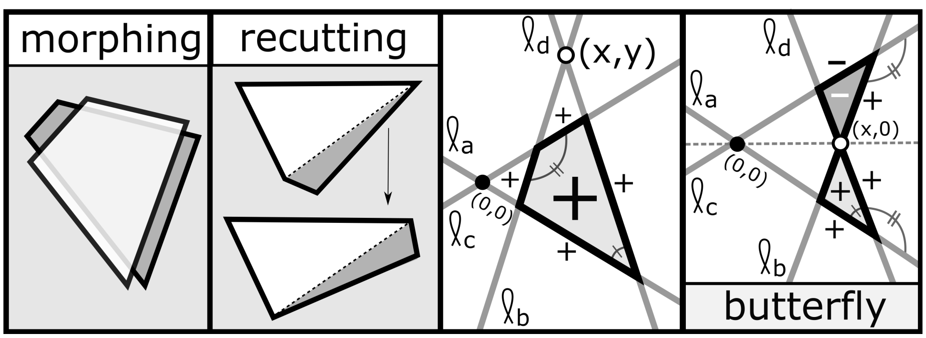
A *supplementary quad* is a convex quadrangle whose opposite interior angles sum to π. Equivalently, the vertices are co-circular. These notes succinctly prove Brahmagupta's formula for such quads: $A^2 = B^2$, where A is the area and $B^2 = (s-a)(s-b)(s-c)(s-d)$. Here a, b, c, d are the side lengths and $s = (a+b+c+d)/2$.

Proof: Let $C = A^2/B^2$. Let X be the space of supplementary quads. To *morph* a quad is to replace it by one with the same angles. To *recut* a quad is to cut along a diagonal and reverse one triangle. Recutting preserves C. We claim morphing does too. From any point in X we can reach all nearby points by morphs and recuts. (Recut, morph, re-recut to continuously alter one pair of opposite angles; repeat using the other diagonal; morph one final time if needed.) Hence C is constant on X. Since X has squares, $C = 1$.



Proof of Claim: Let $L(\rho, \sigma)$ be the space of lines $\ell_a, \ell_b, \ell_c, \ell_d$ with slopes $\rho, \sigma, -\rho, -\sigma$ and $\ell_a \cap \ell_c = (0,0)$. Parametrize $L \cong \boldsymbol{R}^2$ by $(x,y) = \ell_b \cap \ell_d$. Let a, b, c, d be the *signed* distances between vertices of the associated quads, and let A be the *signed* area. Choose signs so that $a, b, c, d, A > 0$ in a convex case. $B^2(x,y)$ is a degree 4 polynomial as $a(x,y), ..., d(x,y)$ are linear. $A(x,y)$, a sum of determinants of linear functions, is a degree 2 polynomial. If $xy = 0$ the quads are *butterflies*, so $A = 0$; also $|a| = |c|$ and $|b| = |d|$ and $ab = -cd$, so 2 factors of B^2 vanish. Given their degrees, $A(x,y) \propto xy$ and $B^2(x,y) \propto (xy)^2$. Hence $C|_L$ is constant. Our claim follows: A (generic) quad and its morphs are all isometric to quads in the same L. ♠

Discussion: John Conway long sought a simple and beautiful *geometric* proof of Brahmagupta's formula, like Sam Vandervelde's recent proof [4]. My proof would not have satisfied Conway; hence my title.

In addition to being short, my proof is pretty elementary. It only depends on basic facts about polynomials and continuity. However, I found this proof by thinking about some deeper modern mathematics. (It also helps to have smart friends; see the acknowledgements at the end.)

The main idea is that the roots, counted with multiplicity, determine a real polynomial up to constants provided that the number of roots equals the degree of the polynomial. A single evaluation then determines the constant. My proof can be summarized like this: The function $C = A^2/B^2$ is invariant under recutting. It is also invariant under morphing because, when analytically continued, A^2 and B^2 vanish to the same order on the set of butterflies and nowhere else. The recutting/morphing process spreads the constancy of C through X like a virus.

To make this idea work, we have to enlarge the space X so that it includes some nonconvex quads, especially butterflies. We'll explain it from another point of view here. First of all, let us modify X so that we consider supplementary quads modulo isometry. Call quads *cousins* if they are morphs of each other. We think of X as a fiber bundle, where the fibers are the cousin families. Each fiber is a convex cone in $\boldsymbol{R}^2$. We create a new space X^* by replacing these cones by the copies of $\boldsymbol{R}^2$ which contain them. The space X^* is a plane bundle with the same base. The fibers are our L spaces.

The fact that A is a degree 2 polynomial on the fibers is a key idea of Bill Thurston's paper *Shapes of Polyhedra* [3] In Thurston's work, he introduces local complex linear coordinates on the space of flat cone spheres with prescribed cone angles. Prescribing the cone angles is like restricting to a fiber. Thurston's coordinates are like my (x, y) coordinates. He shows the area of a flat cone surface with fixed cone angles is the diagonal part of a Hermitian form in his coordinates. There is also a real valued version of this theory which is even closer to my proof, exposited recently in the A.M.S. *Notices* [1] by Danny Calegari. The same ideas also arise in translation surfaces.

The main point is that if you fix the slopes of the lines (or the cone angles, in Thurston's case), various algebraic functions are simplified and become linear. For example, if we fix the angles of a right triangle then its area is a quadratic function in the length of the hypotenuse because both the base and height are linear functions of the length of the hypotenuse. The constant of proportionality α depends on the angles, of course.

Incidentally, the fact just mentioned is the basis for one of the greatest ever proofs of the Pythagorean Theorem. Start with a right triangle T_c with sides a, b, c and hypotenuse c. Drop the altitude to c and consider the two new triangles T_a and T_b whose union is T_c. All three triangles are in the same similarity class. The area of T_c is the sum of the areas of T_a and T_b. But then $\alpha c^2 = \alpha a^2 + \alpha b^2$. Cancelling α we get the result. My proof of Brahmagupta's formula is a close cousin (no pun intended) of this proof.

Now we discard X^* and go back to X. Once we know that C is fiberwise constant on X how do we equate the constants *across* the fibers? Let me mention two alternate approaches. Each fiber contains triangles (i.e. degenerate quads) and then $C = 1$ by Heron's formula. Heron's formula is a degenerate version of Brahmagupta's formula which is somewhat easier to prove, but you would still need to prove it. Better yet, Peter Doyle noticed that each fiber contains a (perhaps nonconvex) quad whose diagonals are perpendicular, and that for such quads Brahmagupta's formula can be verified with some clever but ultimately easy algebra. I'll leave this as a challenge for the interested reader. I like Peter's endgame, but I wanted something with no computation.

My inspiration for the morphing/recutting proof came from control-theory flavored proofs of ergodicity. The prototypical example is E. Hopf's proof [2] that the geodesic flow on a hyperbolic surface is *ergodic*, meaning that any invariant (measurable) function is (almost everywhere) constant. The connection between our modest recutting trick and Hopf's ergodicity proof may seem far-fetched, but consider the picture.

The geodesic flow lives on a 3-manifold, the unit tangent bundle of the surface. This 3 manifold has 2 invariant codimension 1 foliations, the *stable foliation* and the *unstable foliation*. The first step in Hopf's proof is that to use the expansion/contraction properties of the flow along the leaves of the foliations to establish the (almost everywhere) constancy on each leaf of the stable foliation and each leaf of the unstable foliation. The next step is to walk around, going from a stable leaf to an unstable leaf to a stable leaf, etc, to spread this constancy around over the whole 3-manifold.

The space X has 2 codimension 1 foliations, each consisting of quads sharing a pair of opposite interior angles. Morphing and recutting, using one diagonal and then the other, we take a similar kind of walk through the leaves. When we view these 2 foliations as living in the 3-manifold of similarity classes of supplementary quads, this looks a lot like the ergodicity proof.

I can't resist mentioning another way to do the control theory part of the proof. You can algorithmically change any supplementary quad into a square by a finite sequence of morphs and recuts, so $C = 1$. To make the algorithm as clean as possible we note that both morphing and recutting extend to degenerate supplementary quads – i.e. triangles with one marked point. *Morph so as to maximize the intersection angle between the diagonals, recut along the longest diagonal, repeat until done.* The algorithm produces a finite number of marked triangles, then one or two quads with perpendicular diagonals, then a square. To appreciate what this is doing, restrict it to quads inscribed in the unit circle, and note how it increases the minimum length of the diagonals.

Acknowledgements: Peter Doyle rekindled my interest in Brahmagupta's formula by showing me Sam Vandervelde's proof. Then Peter went on to explain how one can rotate a supplementary quad so that it is a quad of $L(\rho, \sigma)$. Something about this rang a bell, and once I realized that this was just like Thurston's paper the rest fell into place. Peter's insight about the connection to $L(\rho, \sigma)$ was my main inspiration. After I explained my proof to Jeremy Kahn, he suggested the idea of taking the quotient by translation and working in $\boldsymbol{R}^2$ rather than $\boldsymbol{R}^4$. This simplified the algebra. Danny Calegari suggested adding diagrams, and Dan Margalit had helpful expository suggestions. This work is supported by my NSF grant DSM-2102802.

References

[1] D. Calegari, *Pentagonum Pentagonorum*, A.M.S. Notices, Sept 2021

[2] E. Hopf, *Ergodic theory and the geodesic flow o surfaces of constant negative curvature*, B. A. M. S. 77(6), (1971) pp 863-877

[3] W. Thurston, *Shapes of polyhedra and triangulations of the sphere*, Geometry and Topology Monographs, Volume 1: The Epstein Birthday Schrift, pp 511-549

[4] S. Vandervelde, *In Search of Conway's Proof*, preprint 2022