The Isoperimetric Inequality

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The classic isoperimetric inequality says that a circle uniquely encloses the region of largest area in the plane for its length. This result has many proofs. I'll give several that don't involve calculations. In some sense I thought of these myself, but some of the classic proofs inspired me. Possibly these proofs are out there somewhere already.

1 Symmetrized Polya Proof

Let S_n denote the space of unit length polygons with $2n$ (possibly repeated) vertices, modulo isometries. Given $P \in S_n$ let $A(P)$ be the area of the union of the bounded components of $\boldsymbol{R}^2 - P$. Let P' be the convex $2n$-gon we get by taking the convex hull of P, padding with extra vertices if necessary, then rescaling to get the length back to 1. Then $A(P') > A(P)$ unless $P = P'$. The function A varies continuously on the compact subset $\Sigma_n \subset S_n$ consisting of convex $2n$-gons. Hence A achieves a maximum on S_n at some $P \in \Sigma_n$. By moving any repeated vertices onto the interiors of other edges, we can assume that P has no repeated vertices and hence $2n$ nontrivial edges.

If P has 2 consecutive edges of different lengths we can shorten P while keeping the area the same by the move shown in Figure 1. Hence, all $2n$ sides of P have the same length.

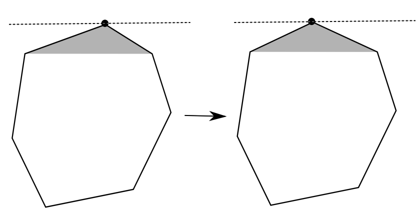

Figure 1: Decreasing the length by sliding parallel to the diagonal

The chord L of P connecting vertices P_0 and P_n divides P into two arcs Q, R of equal length. We can label so that $A(Q \cup L) \geq A(P)/2$. We translate so that the midpoint of L is the origin. Let $P' = Q \cup (-Q)$. By construction $\ell(P') = 1$ and $A(P') \geq A(P)$ and $-P' = P'$. Since P is an area maximizer, we must have $A(P') = A(P)$. In short, we can assume without loss of generality that $-P = P$. Precisely, $P_{k+n} = -P_k$ for all k.

If we fix the side lengths of a parallelogram but allow the angles to vary, the area maximizer in the family is the rectangle. Consider the parallelogram $Q = P_0 P_k P_n P_{k+n}$. Suppose Q is not a rectangle. We divide the region bounded by P into 5 pieces: the region bounded by Q and 4 flaps made from arcs of P and sides of Q. See Figure 2. (We think of P as having so many vertices that the shown arcs are really polygonal.)

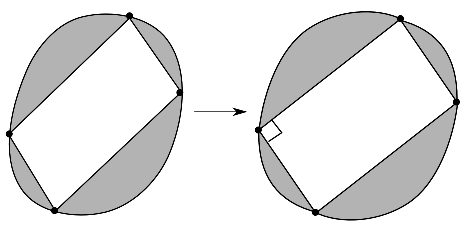

Figure 2: Increasing the area of Q by flexing.

We flex Q into a rectangle and let the flaps move isometrically along for the ride. This increases $A(P)$ and keeps $\ell(P) = 1$, a contradiction. Hence Q is a rectangle. Hence $\|P_k\| = \|P_0\|$. Since this is true for all k, and since all the sides of P have the same length, we see that P is regular. In other words, the regular $2n$-gon is the maximizer. There is a bit more to say: If we started with a non-regular maximizer we could have chosen the symmetrizing chord L so that the symmetrized version was non-circular. This contradiction shows that the only maximizer is the regular $2n$-gon.

Let γ be loop of length 1 and let C be the circle of length 1. We can inscribe $2n$-gons γ_n in γ so that $\lim_{n\to\infty} A(\gamma_n) = A(\gamma)$. From the polygonal case discussed above, $A(\gamma_n) \leq A(C_n)$ where C_n is the regular $2n$-gon of length 1. So

$$A(\gamma) = \lim_{n\to\infty} A(\gamma_n) \leq \lim_{n\to\infty} A(C_n) = A(C).$$

Hence C maximizes area enclosed for length 1 loops. The same argument as above rules out another maximizer γ. We symmetrize so that $-\gamma = \gamma$ and then we do the parallelogram trick.

2 Recutting Proof

Let P be a convex polygon with no repeated vertices. A *k-semidiagonal* of P is a segment that joins the midpoints of two edges of P which are separated by k clicks. Figure 3 shows a 3-semidiagonal. We insist that the cut edges are neither one parallel to L, so that the angles a, b lie in $(0, \pi)$. (The parallelism could happen if P is not strictly convex.) We call the semidiagonal L *isosceles* if $a + b = \pi$. Geometrically this means that the lines extending the cut edges are either both perpendicular to L or else make an isosceles triangle with L.

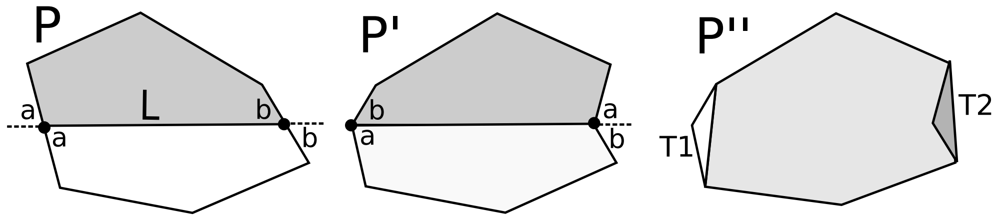

Figure 3: Recutting along a 3-semidiagonal.

The polygon P' in Figure 3 is the union of the lower half of P and the reflection of the upper half of P in the perpendicular bisector of L. Using the notation above, we have $\ell(P') = \ell(P)$ and $A(P') = A(P)$. The two little triangles T_1 and T_2 are isometric; a glide reflection in L carries one to the other. The polygon $P'' = (P' - T_1) \cup T_2$ therefore satisfies $A(P'') = A(P')$ and, by the triangle inequality, $\ell(P'') < \ell(P')$. So, if L is not an isosceles semidiagonal then we can produce P'', having the same number of vertices, such that $A(P'') = A(P)$ and $\ell(P'') < \ell(P)$.

If the 1-semidiagonals of P are all isosceles then all edges of P have the same length. If all edges of P have the same length and all 2-semidiagonals are isosceles then all the interior angles of P are equal and hence P is regular. But, from the construction above, we see that this must happen for an n-gon of length 1 which maximizes area enclosed. This proves that the n-gons which maximize enclosed area for the given length are regular. This works for n both odd and even.

The same limiting argument as above now shows that the circle C is an area maximizer amongst all loops of length 1.

Suppose that γ is another maximizer. We first symmetrize as above so that $-\gamma = \gamma$. We can define recutting for convex loops just as for polygons. Starting with γ and a chord L of γ we let γ' be the union of one side of γ and the reflection of the other side in the perpendicular bisector of L. Note that γ' must be convex; otherwise we could take the convex hull of γ' and produce a new loop γ'' with $A(\gamma'') > A(\gamma)$ and $\ell(\gamma'') < \ell(\gamma)$.

Consider recuttings of γ along diameters of γ. For these, the convexity of γ' implies that γ is contained in the strip bounded by the two perpendiculars to L at $\gamma \cap L$, as shown in Figure 4.

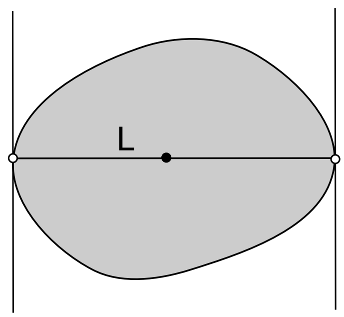

Figure 4: γ is trapped in a strip.

But if γ has the strip property in every direction, it means that the norm of points on γ, as a function of arc length, cannot increase to first order at any point. But this implies that the norms of points on γ cannot increase at all as we move around. Hence γ is a circle.