Notes on Manifolds

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April 22, 2024

1 Introduction

These are some notes that I handed out during the two times I taught (what was) Math 1140 at Brown University. This is a course on manifolds. The course covers the following topics:

1. Facts about differentiation; inverse and implicit function theorems
2. Basic definitions of manifolds; tangent spaces; diffeomorphisms
3. Exterior algebra and differential forms
4. Stokes' Theorem on manifolds and applications.

I used the book *Mathematical Analysis* by Andrew Browder, and mostly covered chapters 11,12,13,14. I often found that the proofs in the book were not as efficient as I would like, so I often wrote up my own notes. I have gathered all these notes together in one place in case someone else teaching a similar course would find them useful. The notes do not cover everything in the book; just topics that I thought I could clarify. Some of the notes are on slightly extraneous topics.

The notes are not guaranteed to be correct! I have tried my best to get everything right but perhaps there are still some glitches and omissions. In particular, you will probably find a lot of typos. Sometimes the proofs are things I thought of myself, but I really don't make many claims to originality. I am sure that I learned practically everything in this bundle somewhere.

Here is a list of topics covered in these notes.

- Two results about differentiation
- Equality of the Mixed Partials
- The Inverse and Implicit Function Theorems
- Elementary Properties of Volume
- Change of Variables formula (for integration in $\boldsymbol{R}^n$)
- Abstract Manifolds: Basic Definitons
- Tangent Spaces and Orientation
- Tensor Transformations
- Partitions of Unity
- The Poincare Lemma (for deRham cohomology)
- The Brouwer Fixed Point Theorem
- Integrating Functions on Manifolds
- Harmonic Functions and the Hodge Star Operator

Each topic is contained in an essentially stand-alone set of notes. However, occasionally the later notes refer back to the earlier ones. I have tried to minimize this.

2 Two Results about Differentiation

The purposes of these notes is to prove two results about differentiation. The first result is that C^1 maps are differentiable and the second result is the Chain Rule. We assume that all maps we consider fix the origin. You can easily extract the general case from this.

2.1 Basic Definitions

Let $F : \boldsymbol{R}^m \to \boldsymbol{R}^n$ be a map. Let $e_1, ..., e_m$ be the standard basis vectors. The expression

$$\frac{\partial F}{\partial x_j}(a) = \lim_{h \to 0} \frac{F(a + he_j) - F(a)}{h} \tag{1}$$

is called the jth partial derivative of F at a. The higher partial derivatives are defined in an iterative way. For instance

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\Big(\frac{\partial F}{\partial x_j}\Big).$$

The derivative just listed has order 2. In general, the partial derivative of a kth order partial derivative has order $k + 1$.

Here are some basic definitions.

1. F is called C^k if the order j partial derivatives exist and are continuous for all $j = 1, ..., k$.

2. F is called C^∞ if F is C^k for all k. In this case, F is also called *smooth*.

In this class we are going to work exclusively with smooth maps.

It turns out that when F is C^k, all the defined partial derivatives commute. In particular, if F is C^2 then

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}.$$

This is by no means obvious. Your first homework assignment takes you through a proof.

Here is a related notion. Now assume that $F(O) = O$, where O is the origin. We say that F is *differentiable* at O if there is a linear transformation T with the following property.

$$\lim_{\|v\|\to 0} \frac{\|F(v) - T(v)\|}{\|v\|} = 0 \tag{2}$$

To expand this out, Equation 2 says that for any $\epsilon > 0$ there is some N such that $\|v\| < 1/N$ implies that $\|F(v) - T(v)\| < \epsilon\|v\|$. We usually denote T by F^*. When F is differentiable at 0, the first partial derivatives exist and

$$\frac{\partial F}{\partial x_j} = F^*(e_j). \tag{3}$$

This means that F^* is just the matrix of first partial derivatives evaluated at the origin.

The converse is trickier. It is not obvious that F is differentiable at O even if all its partial derivatives exist at all points. In fact, this is false in general.

2.2 A Strange Example

Here we construct a map $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ which has partial derivatives at all points of $\boldsymbol{R}^2$ but is not differentiable at O.

Choose any smooth $(\pi/2)$-periodic function ϕ, with $\phi(0) = 0$. This means that $\phi(k\pi/2) = 0$ for $k = 1, 2, 3, ...$ We arrange that $\phi(\pi/4) = 1$. Using polar coordinates, define $F(r, \theta) = r\phi(\theta)$. Here are some properties of F:

- F is continuous on $\boldsymbol{R}^2$ and C^∞ on $\boldsymbol{R}^2 - O$.
- The partial derivatives of F exist everywhere. Since $F = 0$ on the coordinate axes we have $\partial F/\partial x_1(O) = \partial F/\partial x_1(O) = 0$.

Suppose F is differentiable at O. The polar coordinates of the vector (h, h) are $(h\sqrt{2}, \pi/4)$. Hence $F(h, h) = h\sqrt{2}$. If F is differentiable at O then F^* is the 0 matrix, because it is given by the matrix of first partials. Thus $F^*(h, h) = 0$ for all h. In particular $F^*(1, 1) = 0$. On the other hand, by Equation 2 we have

$$F^*(1,1) = \frac{1}{h}F^*(h,h) = \lim_{h\to 0} \frac{F^*(h,h)}{h} = \lim_{h\to 0} \frac{F(h,h)}{h} = \sqrt{2}.$$

This is a contradiction. Hence F is not differentiable at O. The issue here is that F is not C^1.

2.3 Continuous Differentiability

You don't have to worry about garbage like the above example in our class. In this section we prove that F is differentiable at O provided that F is C^1. This result extends easily to other points. So, C^1 maps are differentiable everywhere in their domain. In particular, smooth maps are differentiable at every point in their domain and their derivative matrices vary continuously.

Let us call a function F *nice* if F is C^1 and differentiable at O. We want to prove that all C^1 functions (which fix O) are nice.

Let us first clean off the statement we want to prove. Here F is a general map from $\boldsymbol{R}^m$ to $\boldsymbol{R}^n$. We can write $F = (F_1, ..., F_n)$. The function F is nice if and only if each F_i separately is nice. So, it suffices to prove the case when $F : \boldsymbol{R}^m \to \boldsymbol{R}$ is a function.

We now observe that two functions G_1 and G_2 are both nice, then so is $G_1 \pm G_2$. This follows from the usual sum and difference rules for taking limits. We next observe that all linear functions are nice. In particular, the matrix F' of partials of F (which is really just the gradient of F) at the origin is also nice. This means that $F - F'$ is nice if and only if F is nice. So, we can assume that all $\partial F/\partial x_i(O) = 0$ for all $i = 1, ..., m$.

Given a point $v = (x_1, ..., x_n)$ let $v_k = (x_1, ..., x_k, 0, ..., 0)$. Note that v_k and v_{k+1} differ in the $(k+1)$st coordinate. Notice that

$$F(v) = \sum_{k=1}^{n-1} (F(v_{k+1}) - F(v_k)).$$

Hence

$$\|F(v)\| \leq \sum_{k=1}^{n-1} \|F(v_{k+1}) - F(v_k)\|. \tag{4}$$

Let L_k be the line segment connecting v_{k+1} to v_k. The restriction of F to L_k is just a single variable function. This single variable function is, in particular, differentiable.

Lemma 2.1 $\|F(v_{k+1}) - F(v_k)\| \leq A_k B_k$ *where*

$$A_k = \|v_{k+1} - v_k\|, \qquad B_k = \sup_{q \in L_k} \left\| \frac{\partial F}{\partial x_{k+1}}(q) \right\|.$$

Proof: This is just the Fundamental Theorem of Calculus. But let's give a self-contained proof. By scaling and translation it suffices to consider the single variable case of $f : \boldsymbol{R} \to \boldsymbol{R}$ with $f(0) = 0$ and $\sup_{x\in[0,1]} |f'(x)| \leq 1$. We want to see that $|f(1)| \leq 1$.

Fix some $\eta > 1$. Let us call a sub-interval $J \subset [0,1]$ *bad* if

$$|f(J_0) - f(J_1)| > \eta |J|.$$

Here J_0, J_1 are the endpoints of J and $|J|$ is the length of J. We are trying to prove that $[0,1]$ is not bad with respect to any $\eta > 1$.

If J is bad then one of the two intervals obtained by subdividing J in half is bad. This is just the triangle inequality. So, if $[0,1]$ is bad we can find an infinite nested sequence $\{J_n\}$ of intervals, all bad, such that $|J_n| \to 0$. By compactness (or the completeness of $\boldsymbol{R}$) the intersection $\bigcap J_n$ has a single point x of intersection. But, by construction $|f'(x)| \geq \eta$. This is a contradiction. ♠

Now we apply Lemma 2.1 to Equation 4. Since F is C^1 and the partials vanish at 0, we can make these partials as small as we like when we work close to the origin. That is, we can make the expression B_k as small as we like by taking $\|v\|$ small. But then, for any $\epsilon > 0$ we can find some N such that when $\|v\| < 1/N$ we have $B_k \leq \epsilon$ for all k. At the same time we have $A_k < \|v\|$ for all k. Hence, by Equation 4 $\|F(v)\| \leq n\epsilon\|v\|$. But then

$$\frac{\|F(v) - T(v)\|}{\|v\|} < n\epsilon.$$

Here T is the 0-map. This shows that F is differentiable at O and F^* is the 0-map. Hence F is nice.

This completes the proof.

2.4 Reformulation of Differentiability

Our next goal is to prove the Chain Rule. We first reformulate the notion of differentiability and then give the Chain Rule Proof.

Let $F : \boldsymbol{R}^m \to \boldsymbol{R}^n$ be a map such that $F(O) = O$. Let D_n be the map which dilates distances by $1/n$. For n large, D_n is massively shrinking points. Define

$$F_n = D_n \circ F \circ D_{1/n}. \tag{5}$$

Note that F_n is differentiable at 0 if and only if F is differentiable at 0. Also $F_n^* = F^*$. Finally, the behavior of F on any set K is the same as the behavior of F_n on the set $D_n(K)$. We call this the *scaling principle*.

Definition: Let G_n be a family of functions indexed by the positive reals. We say that $G_n \to G$ *uniformly on a compact set* K if the following is true. For any $\epsilon > 0$ there is some N such that $n > N$ implies that $\|G_n(x) - G(x)\| < \epsilon$ for all $x \in K$. We say that $G_n \to G$ *uniformly on compacta* if $G_n \to G$ uniformly on any compact set K.

Now we reformulate differentiability in terms of this kind of convergence.

Lemma 2.2 *Suppose F is differentiable at O. Then $F_n \to F^*$ uniformly on compacta.*

Proof: Let K be an arbitrary compact subset of $\boldsymbol{R}^m$. Being compact, K is closed and bounded. So, we can assume that K is the ball of some radius centered at the origin. Using the scaling principle (i.e., by replacing F by some F_n if needed) we can assume that K is the unit ball.

Let $\epsilon > 0$ be given. Equation 2 says that there is some N such that $\|v\| < 1/N$ implies that

$$\|F(v) - F^*(v)\| < \epsilon\|v\|.$$

In particular this is true for the vector $v = w/N$, where $w \in K$ is any vector. What we are saying is that

$$\|F(w/N) - F^*(w/N)\| < \epsilon\|w/N\| \leq \epsilon/N.$$

But this is just saying that

$$\|F \circ D_{1/N}(w) - F^* \circ D_{1/N}(w)\| \leq \epsilon/N.$$

Scaling this equation by N we find that

$$\|F_N(w) - F^*(w)\| < \epsilon. \tag{6}$$

Here we are using the fact that $D_N \circ F^* \circ D_{1/N} = F^*$. Equation 6 exactly expresses the uniform convergence condition. ♠

Lemma 2.3 *If $F_n \to T$ uniformly on compacta then F is differentiable at* 0 *and $T = F^*$.*

Proof: The proof really just amounts to reversing the implications of all the steps given for the previous proof. Let's work it out. The hypotheses imply, in particular, that $F_n \to T$ uniformly on the unit ball.

Let $\epsilon > 0$ be given. There is some N so that $\|F_n(w) - T(w)\| < \epsilon$ for any unit vector w in the ball of radius 1 centered at the origin and any $n \geq N$. If we have any vector v with $\|v\| \leq 1/N$ then we can write $v = w/n$ for some unit vector w and some $n \geq N$. (This n is not necessarily an integer.) We have

$$\|F(v) - T(v)\| = \|F(w/n) - T(w/n)\| =$$
$$\|D_{1/n} \circ F_n(w) - D_{1/n} \circ T(w)\| \leq \epsilon/n = \epsilon\|v\|.$$

Dividing through by $\|v\|$ we see that

$$\frac{\|F(v) - T(v)\|}{\|v\|} < \epsilon$$

provided that $\|v\| < 1/N$. This is equivalent to differentiability. By defintion, we have $T = F^*$. ♠

2.5 The Chain Rule

Now suppose that $H = F \circ G$. We assume that all these functions map 0 to 0. We also assume that F and G are differentiable at 0. We want to prove that H is differentiable at 0 and that $H^* = F^* \circ G^*$. We will use the reformulation.

Recall that $F_n = D_n \circ F \circ D_{1/n}$. Define G_n and H_n similarly. We have

$$H_n = F_n \circ G_n. \tag{7}$$

Let K be any compact set and let $v \in K$. We have

$$\|H_n(v) - F^* \circ G^*(v)\| =$$
$$\|F_n \circ G_n(v) - F^* \circ G^*(v)\| =$$
$$\|F_n \circ G_n(v) - F^* \circ G_n(v) + F^* \circ G_n(v) - F^* \circ G^*(v)\| \leq$$
$$\|F_n \circ G_n(v) - F^* \circ G_n(v)\| + \|F^* \circ G_n(v) - F^* \circ G^*(v)\|.$$

Call the terms on the last line A and B. We estimate these two terms separately. Because $G_n \to G$ uniformly on compacta, there is some larger compact subset K' such that $G_n(K) \subset K'$ for n sufficiently large. But $F_n \to F^*$ uniformly on K'. That means that we can make A as small as we like, independent of v, by taking n large enough.

Consider B. The map F^*, being a linear transformation, only expands distances by at most a constant factor: There is some ℓ with the following property: If $\|w_1 - w_2\| < \epsilon$ then $\|F^*(w_1) - F^*(w_2)\| < \ell\epsilon$. We can make $\|G_n(v) - G^*(v)\|$ as small as we like by taking n large. But then B will only be ℓ times bigger. Hence, we can make B as small as we like by taking n large enough.

Since we can make both A and B as small as we like, we can make $\|H_n(v) - F^* \circ G^*(v)\|$ as small as we like by taking n large. This shows that H_n converges uniformly on compacta to $F^* \circ G^*$. But this means that H is differentiable at 0 and $H^* = F^* \circ G^*$.

3 Equality of the Mixed Partials

This set of notes is actually a worksheet. The goal of the worksheet is to prove that the mixed second partials of a function $F : \boldsymbol{R}^n \to \boldsymbol{R}$ are equal when they are continuous. This seems to be one of the most fundamental results about partial derivatives.

Mostly we'll work with $n = 2$. One disadvantage to this outline is that the first problem is possibly the hardest.

Defintion: Say that $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ is *special* if F has continuous second partial derivatives, and F vanishes on the coordinate axes. That is, $F(t,0) = F(0,t) = 0$ for all t.

1. Suppose F is special and

$$\frac{\partial^2 F}{\partial x \partial y}(0,0) = 0.$$

Prove that

$$\lim_{t \to 0} \frac{F(t,t)}{t^2} = 0. \tag{8}$$

Hint: First show that

$$|F(t,t)| \leq t \times \sup_{s \in [0,t]} |\partial F / \partial y(t,s)|.$$

2. Suppose that F is special and

$$\frac{\partial^2 F}{\partial x \partial y}(0,0) = C.$$

$$\lim_{t \to 0} \frac{F(t,t)}{t^2} = C. \tag{9}$$

(Hint: Apply Exercise 1 to the function $G(x,y) = F(x,y) - Cxy$, which is again special.

3: Prove that the second mixed partials of a special function are equal at the origin. Hint: use Exercise 2.

4. Let $\mathcal{V}$ denote the set of functions on $\boldsymbol{R}^2$ whose second mixed partials exist and are equal at the origin. Prove that $\mathcal{V}$ is a real vector space. (The addition law is just $(f+g)(x) = f(x)+g(x)$). It follows from Exercise 3 that $\mathcal{V}$ contains all special functions.

5. Say that a function $G : \boldsymbol{R}^2 \to \boldsymbol{R}$ is *simple* if one of the following two properties holds:

- $G(x, y) = F(x, 0)$ for some function F having first partial derivatives.
- $G(x, y) = F(0, y)$ for some function F having first partial derivatives.

Prove that the second mixed partials of a simple function are 0. Hence, the vector space $\mathcal{V}$ contains all simple functions, and all finite sums of simple functions.

6. Let $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function whose second partials exist and are continuous. Prove that F is the sum of a special function and finitely many (in fact three) simple functions. Hence $F \in \mathcal{V}$.

7. Let $F : \boldsymbol{R}^n \to \boldsymbol{R}$ be a function whose second partials exist and are continuous. Prove that

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i},$$

for all i and j. Hint: reduce this to the case $n = 2$, and then compose F with suitable translations. In other words, if you can prove something at the origin for all functions, you can prove the same thing for all functions at all points.

4 Inverse and Implicit Function Theorems

The purpose of these notes is to prove the Inverse Function Theorem and the Implicit Function Theorem.

4.1 Some Technical Preliminaries

If you have a smooth function F and dF is invertible, you can translate, compose with linear transformations, and scale so that dF is really close to the identity in a huge neighborhood. Let's study this situation first.

Let B_r denote the ball of radius r centered at the origin. Suppose that F is defined in B_{100}, and smooth, and at all points F^* is within 10^{-100} of the identity matrix. The first result is crucial to the whole business. It says that the vectors $F(p) - F(q)$ and $p - q$ are almost the same vector in some sense.

Lemma 4.1 *Given any points $p, q \in B_{100}$, we have*

$$\|(F(q) - F(p)) - (p - q)\| < \|p - q\|/1000.$$

Proof: Let $\gamma_1 : I \to \boldsymbol{R}^n$ be the straight line segment joining p to q. We parametrize so that γ_1 has unit speed. Let $\gamma_2 = F(\gamma_1)$. By the Chain Rule $d\gamma_2/dt = F^*(d\gamma_1/dt)$. Given that F^* is within 10^{-100} of the identity along γ_1, we see that $d\gamma_1/dt$ and $d\gamma_2/dt$ are almost the same vector at each point. More precisely, $d\gamma_2/dt = d\gamma_1/dt + v$, where $\|v\| < 1/1000$. Integrating, we see that

$$(F(q) - F(p)) - (q - p) = \int_I d\gamma_2/dt\ dt - \int_I d\gamma_2/dt\ dt = \int_I v\ dt.$$

This last vector-valued integral has norm less than $\|p - q\|/1000$. ♠

Corollary 4.2 *F is injective on B_{100}.*

Proof: Suppose not. Then we have $p \neq q \in B_{100}$ with $F(q) - F(p) = 0$. This violates Lemma 4.1. ♠

Let O be the origin. Assume now that $F(O) = O$.

Lemma 4.3 *$B_1 \subset F(B_{10})$.*

Proof: Suppose this is false. Let $y \in B_1$ be a point not in the image of B_{10}. Let $\phi(x) = \|F(x) - y\|$. Let

$$\alpha = \inf_{x \in B_{10}} \phi(x).$$

Since B_{10} is compact there is some $x \in B_{10}$ such that $\phi(x) = \alpha$. If $\alpha = 0$ we are done. Suppose not.

By Lemma 4.1, we have $\|x\| < 2$. Let $x' = x + (y - F(x))$. We have $x' \in B_4 \subset B_{10}$. By Lemma 4.1, there is some v with $\|v\| \leq \alpha/100$ such that

$$F(x') - F(x) = x' - x + v = y - F(x) + v.$$

Simplifying this, we get $F(x') - y = v$. Hence $\phi(x') < \alpha$, a contradiction. ♠

We need one last technical result.

Lemma 4.4 *Let $\phi = g \circ h$ where g is smooth and h is k-times differentiable. Then ϕ is k-times differentiable.*

Proof: This goes by induction. Let f^* be the matrix derivative of f, and likewise for the other functions. When $k = 1$ the result is just the Chain Rule. Consider the general case. From the Chain Rule, we have

$$\phi^* = (g^* \circ h) \times h^*.$$

The product is matrix multiplication. The function g^* is smooth because g is smooth. The function h is $k - 1$ times differentiable because it is (more strongly) k times differentiable. So, by induction $g^* \circ h$ is $k - 1$ times differentiable. Also h^* is $k - 1$ times differentiable because it is the matrix of first partials of h. By the product rule, this product is also $k - 1$ times differentiable. Since ϕ^* is $k-1$ times differentiable, ϕ is k-times differentiable. ♠

4.2 The Inverse Function Theorem

Let U, V be subsets of $\boldsymbol{R}^n$. A *diffeomorphism* from U to V is a bijection $F : U \to V$ such that F and F^{-1} are both smooth and the derivatives F^* and $(F^{-1})^*$ are invertible at all points of their domains.

Here is the Inverse Function Theorem.

Theorem 4.5 *Let U be an open subset of $\boldsymbol{R}^n$ and let $f : U \to \boldsymbol{R}^n$ be a smooth map. Suppose that f^* is invertible at some point $x \in U$. Then there are open subset $U_x \subset U$ and $V_x = F(U_x)$ such that $x \in U_x$ and $F : U_x \to V_x$ is a diffeomorphism.*

Composing with linear transformations, translating, and scaling, we can assume that F is normalized as in §4.1. The results §4.1 show that F is injective on B_{100}. In particular, F is injective on U, the open unit ball.

Lemma 4.6 *V is an open set and F^{-1} is continuous on V.*

Proof: Choose any $y \in V$ and let $x \in U$ be such that $F(x) = y$. By translation and scaling and Lemma 4.3, F maps some small ball around x to a set which contains a small ball around y. Hence V contains an open ball centered about y. Hence V is open

To show that F^{-1} is continuous it suffices to show that F maps open subsets of U to open subsets of V. But this is just the same argument that we just gave. ♠

Lemma 4.7 *F^{-1} is differentiable and $(F^{-1})^* = (F^*)^{-1} \circ F^{-1}$ on V.*

Proof: Let D_n be dilation by n. Translating, we reduce to proving this equation at O. The dilated map $F_n = D_n \circ F \circ D_{1/n}$ converges to $F^*|_O$ uniformly on compacta. But $(F^{-1})_n = (F_n)^{-1}$, and $(F_n)^{-1}$ converges uniformly on compacta to $(F^*)^{-1}$. But this implies that F^{-1} is differentiable at O and its derivative is the inverse of $F^*|_O$. ♠

Now we know that F^{-1} is differentiable. Suppose that F^{-1} is k-times differentiable. The formula in Equation 4.7 combines with Lemma 4.4 to show that $(F^{-1})^*$ is k-times differentiable. Hence F^{-1} is $k+1$ times differentiable. By induction, F^{-1} has partial derivatives of all orders. Since differentiable functions are continuous, all the partials of F^{-1} are also continuous. Hence F^{-1} is smooth. We're done.

4.3 The Implicit Function Theorem

Suppose that $F : \boldsymbol{R}^N \to \boldsymbol{R}^n$ with $n < N$. We call $p \in \boldsymbol{R}^N$ a *regular point* for F if $F^*|_p$ is a surjective linear map. Let $F|_V$ be the restriction of F to an open subset V.

Theorem 4.8 (Implicit Function Theorem) *Suppose that p is a regular value for F. Let $q = F(p)$. Then there is an open neighborhood V of p and an an open subset $U \subset \boldsymbol{R}^{N-n}$ and a smooth bijection $f : U \to V \cap (F|_V^)-1(q)$ such that f^* has full rank at each point.*

I am stopping short of calling f a diffeomorhism because it is not a map from a Euclidean space to itself. It is a map from $\boldsymbol{R}^{N-n}$ to $\boldsymbol{R}^N$. Otherwise it behaves like a diffeomorphism.

Composing with linear maps and translating, it suffices to consider the case when $F(O) = O$ and $F^*|_O$ is just the projection from $\boldsymbol{R}^N$ to $\boldsymbol{R}^n$. In particular, $F^*|_O$ is the identity on the vectors $e_1, ..., e_n$ and kills the remaining standard basis vectors.

We introduce the new map $\widehat{F} : \boldsymbol{R}^N \to \boldsymbol{R}^N$ by the formula

$$\widehat{F}(v_1, ..., v_n, v_{n+1}, ..., v_N) = (F(v_1, ..., v_N), v_{n+1}, ..., v_N). \tag{10}$$

That is, the first n coordinates are taken up by F and then we pad out the remaining coordinates. By construction $\widehat{F}^*|_O$ is the identity matrix. By the **Inverse Function Theorem**, There are open subsets $\widehat{U}$ and $\widehat{V}$ about the origin such that $\widehat{F}^* : \widehat{U} \to \widehat{V}$ is a diffeomorphism. We can trim these sets so that $\widehat{V}$ is an open ball centered at the origin. Now let Π be the copy of $\boldsymbol{R}^{N-n}$ given by the last $N - n$ coordinates. That is, $\Pi = \{0\} \times \boldsymbol{R}^{N-n}$. Notice that $\widehat{F}(x) \in \Pi$ if and only if $F(x) = O$. Hence $\widehat{F}$ gives a map from $\widehat{U} \cap (F|_{\widehat{U}})^{-1}(O)$ to $\widehat{V} \cap \Pi$.

Since $\widehat{F}$ is a diffeomorphism from $\widehat{U}$ to $\widehat{V}$, the inverse $\widehat{F}^{-1}$ gives a diffeomorphism from $\widehat{V}$ to $\widehat{U}$, and this diffeomorphism maps $\widehat{V} \cap \Pi$ to the set we care about, $\widehat{U} \cap (\widehat{F}_{\widehat{U}})^{-1}(O)$. Now we change notation in the following way:

- Let $U = \widehat{V} \cap \Pi$.
- Let $V = \widehat{U}$.
- Let f be the restriction of $\widehat{F}^{-1}$ to U.

By construction f is a smooth bijection from U to $V \cap F^{-1}(O)$. Since $\widehat{F}^{-1}$ is a diffeomorphism, the derivative of f has full rank everywhere.

5 Elementary Properties of Volume

This is a worksheet which deals with elementary properties of volume. Problem 5 is the really significant and useful problem.

1: Define the area of a parallelogram P in $\boldsymbol{R}^2$ in the following way. For each covering $\mathcal{U}$ of P by a finite union of squares $Q_1, ..., Q_n$ whose sides are parallel to the coordinate axes define

$$\mu(\mathcal{U}) = \sum_{i=1}^{n} d_i^2, \qquad d_i = \text{side length}(Q_i).$$

Now define

$$\text{area}(P) = \inf_{\mathcal{U}} \mu(\mathcal{U})$$

where the infimum is taken over all covers. Of course this definition works much more generally, and also coincides with the Lebesgue measure of P.

Let M be an elementary matrix, namely one with 1s on the diagonal and then a single other nonzero element. Let Q be the unit square. Prove that $M(Q)$ also has area 1. This matches with the fact that $\det(M) = 1$.

2: Suppose that M is an elementary 2×2 matrix as in Problem 1. Use the result of Problem 3 to prove that for any parallelogram P the two parallelograms P and $M(P)$ have the same area. Prove the same result when M is a diagonal matrix with positive entries.

3: Prove that for every positive determinant 2×2 matrix M and every parallelogram P we have

$$\frac{A(M(P))}{A(P)} = \det(M).$$

Here $A(\cdot)$ is the area function defined as in Problem 3. Hint: Use the fact that M is the product of a diagonal matrix and elementary matrices.

The purpose of this problem is to reconcile two common definitions of the area of a parallelogram. One definition is given by cube covers as above, and the other is just that the area of a parallelogram P is $\det(M)$ where M is positive determinant linear transformation mapping the unit square Q to P.

4: Formulate a result similar the one in in Problem 3 for $\boldsymbol{R}^n$ and at least sketch how you would prove it.

5: Let M be an positive determinant linear transformation of R^n. Suppose that $\{f_j\}$ is a sequence of smooth maps defined on $\boldsymbol{R}^n$ such that $f_j(0) = 0$ and the matrix derivative Df_j differs from M at every point by less than $1/j$. Let Q be the unit cube. Use the result of Problem 6 to show that the volume of $f_j(Q)$ converges to $\det(M)$. Extended Hint: look at the symmetric difference between are $f_j(Q)$ and $M(Q)$. Show this is small. You to this by writing $f_j = L_j + \epsilon_j$ where L_j is the linear map given by $Df_j|_0$ and ϵ_j is a map that you prove is extremely small.

6 The Change of Variables Formula

Consider the following data.

1. U and V are open subsets in $\boldsymbol{R}^n$.
2. $F : U \to V$ is a diffeomorphism.
3. $f : V \to \boldsymbol{R}$ be a continuous function.
4. $K \subset V$ is a compact set.

The purpose of these notes is to give a self-contained proof of the following result.

$$\int_K f \; dV = \int_{F^{-1}(K)} (f \circ F) \; \det(dF). \tag{11}$$

The result also holds when f is just Lebesgue measurable. But, this result requires some auxiliary results from measure theory, like the monotone convergence theorem. The special case when f is continuous suffices for all applications in the class, because these have to do with integrating smooth differential forms on manifolds.

I'll prove the result through a series of steps, each treating a more general case.

6.1 Step 1

The case when K is a cube and F is a linear transformation and f is a constant function just boils down to the determinant.

6.2 Step 2

Let's prove this result when K is a cube and f is a constant function. If $f = 0$ then both integrals are obviously 0. So, we can scale so that $f = 1$. Introduce the function

$$J(K, F) = \frac{\int_{F^{-1}(K)} \det(DF)}{\mu(K)}. \tag{12}$$

Equation 11 is equivalent to the statement that $J(K, F) = 1$.

Suppose that there is some $b > 0$ such that $J(K, F) > 1 + b$. Then, for every $\epsilon > 0$, there is some sub-cube $K' \subset K$ such that the side length of

K' is less than ϵ and $J(K', F) > 1 + b$. This comes from the additivity the integral. If the ratio were near 1 on all small scales, it would also be near one on the large scale.

However, once ϵ is sufficiently small, the restriction of F to K' is nearly a linear map, and the ratio $J(K', F)$ must converge to 1. This is a contradiction. The same argument shows that there cannot be any $b > 0$ so that $J(K, F) < 1 - b$.

These two cases combine to show that $J(K, F) = 1$.

6.3 Step 3

Suppose that K is a cube and f is continuous. This time define

$$J(K, F, f) = \frac{\int_{F^{-1}(K)} (f \circ F) \det(DF)}{\int_K f} \tag{13}$$

The same argument as in Step 2 works here. The point is that the restriction of f to a small cube $K' \subset K$ is nearly constant. So, up to an error which vanishes as $\epsilon > 0$ we are back in the constant function case.

6.4 Step 4

Say that two cubes are *almost disjoint* if they have disjoint interiors. Say that K is *approximable by cubes* if, for every $\epsilon > 0$, there is some finite collection $Q_1, ..., Q_m$ of almost disjoint cubes (with m depending on ϵ) so that

$$K \subset Q = \bigcup_{i=1}^{m} Q_i, \qquad \mu(K) > \sum_{i=1}^{m} \mu(Q_i) - \epsilon. \tag{14}$$

Here μ denotes Lebesgue measure.

Now I'll prove the result assuming that K is approximable by cubes. Once ϵ is sufficiently small, we have $Q \subset V$. By compactness, there is some upper bound C_1 for the restriction of $|f|$ to Q. Hence

$$\left| \int_Q f - \int_K f \right| < C_1 \epsilon. \tag{15}$$

Since F is a diffeomorphism and Q is compact, there exists some constant C_2' such that the restriction of F^{-1} to Q expands distances by a factor of C_2' and hence volume by at most $C_2 = (C_2')^n$. Hence

$$\mu(F^{-1}(Q - K)) < C_2 \epsilon. \tag{16}$$

By compactness again, there is a constant C_3 so that the restriction of $|\det(DF)|$ to $F^{-1}(Q)$ is at most C_3. Hence

$$\left|\int_{F^{-1}(Q)}(f\circ F)\ \det(DF)-\int_{F^{-1}(K)}(f\circ F)\ \det(DF)\right|<C_1C_2C_3\epsilon. \tag{17}$$

Since the cubes $Q_1,...,Q_m$ are almost disjoint, we have

$$\sum_{i=1}^{m}\int_{Q_i}f=\int_{Q}f. \tag{18}$$

$$\sum_{i=1}^{m}\int_{F^{-1}(Q_i)}(f\circ F)\ \det(DF)=\int_{F^{-1}(Q)}(f\circ F)\ \det(DF). \tag{19}$$

Since Equation 11 is true for individual cubes, it is also true for finite sums of cubes, as in the set Q. But then Equations 15 and 17 tell us that Equation 11 holds for K up to an error of $C_4\epsilon$, where $C_4=C_1+C_1C_2C_3$. But ϵ is artibrary. Hence Equation 11 holds for K.

6.5 Step 5

Now we show that every compact $K\subset V$ is approximable by cubes. Without loss of generality, we can assume that $K\subset[0,1]^n$. For notational convenience, set $X=[0,1]^n$. Say that a *dyadic interval* is an interval whose endpoints are rational numbers of the form $k/2^m$ for integers k and m. Say that a *dyadic cube* is the product of dyadic intervals which all have the same length. The set of centers of dyadic cubes is dense in $\boldsymbol{R}^n$ and also the set of possible diameters of such cubes is dense. For this reason, $X-K$ is the countable union of dyadic cubes.

Let $P_1,P_2,P_3,...$ be this infinite collection. We have

$$\sum_{i=1}^{\infty}\mu(P_i)=\mu(X-K). \tag{20}$$

Setting

$$P^{\ell}=\bigcup_{i=1}^{\ell}P_i, \tag{21}$$

we have

$$\lim_{\ell\to\infty}\mu(P^{\ell})=\mu(X-K),\qquad K\subset X-P^{\ell}. \tag{22}$$

Given ϵ, we can choose ℓ so that $\mu(X-K) < \mu(P^\ell) + \epsilon$. Using the fact that

$$\mu(K) + \mu(X-K) = 1 = \mu(P^\ell) + \mu(X - P^\ell) \tag{23}$$

we see that

$$\mu(K) > \mu(X - P^\ell) - \epsilon. \tag{24}$$

But $X - P^\ell$ is a finite union of almost disjoint cubes, say $Q_1, ..., Q_m$. The way to see this is that we can scale up the whole picture by some power of 2 so that every cube in sight has integer coordinates. Then the set of interest to us is tiled by integer cubes.

7 Manifolds

The purpose of these notes is to define what is meant by a *manifold*, and then to give some examples.

7.1 Topological Spaces

If you haven't seen topological spaces yet, just skip this section.

The space underlying a manifold is traditionally taken to be a second-countable Hausdorff topological space. To say that a space X is *second countable* is to say that there is a countable collection of open subsets of X such that every open subset of X is a union of members from the countable collection – i.e., X has a *countable basis*. To say that X is *Hausdorff* is to say that, for every two distinct points $x, y \in X$, there are disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$.

That is all I'm going to say about topological spaces. Below I'm going to define manifolds in terms of metric spaces. The definition I give is equivalent to the definition that is given in terms of topological spaces, even though at first glance it looks different.

7.2 Metric Spaces

A *metric space* is a set X together with a function $d : X \times X \to \boldsymbol{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

d is called the *distance function* on X.

Example 0: It almost goes without saying, but I'll say explicitly that any subset of a metric space is automatically a metric space, with the same metric. This fact is frequently and implicitly used.

Example 1: The classic example of a metric space is a subset $X \subset \boldsymbol{R}^n$ equipped with the distance function given by $d(x, y) = \|x - y\|$, here $\| \cdot \|$ is the Euclidean norm.

Example 2: This example is unrelated to the rest of the material in the notes, but I like it. Choose a prime p and on $\boldsymbol{Z}$ define $d_(x, y) = p^{-k}$, where k is the largest integer such that p^k divides $x - y$. This is known as the p-adic metric on $\boldsymbol{Z}$. Geometrically, $\boldsymbol{Z}$ looks like a dense subset of points in a Cantor set when it is equipped with the p-adic metric.

From now on, X denotes a metric space, and d the metric on X.

Balls: Given $x \in X$ and some $r > 0$, we define

$$B_r(x) = \{y \in X|\ d(x, y) < r\}. \tag{25}$$

The set $B_r(x)$ is known as the open ball of radius r about x.

Open Sets: A subset $U \subset X$ is *open* if, for every $x \in U$, there is some $r > 0$ such that $B_r(x) \subset U$.

Continuity: Given to metric spaces X and Y, a map $f : X \to Y$ is called *continuous* if, for all open $V \subset Y$ the inverse image $U = f^{-1}(V)$ is open in X. This definition is equivalent to the usual $\epsilon - \delta$ definition of continuity. From our definition, it is clear that the composition of continuous functions is continuous. If $f : X \to Y$ and $g : Y \to Z$ are both continuous, then so is $g \circ f : X \to Z$.

Homeomorphisms: A map $f : X \to Y$ is a *homeomorphism* if f is a bijection and both f and f^{-1} are continuous. So, in particular, a homeomorphism from X and Y induces a bijection between the open subsets of X and the open subsets of Y. To test your understanding, prove that the open ball in $\boldsymbol{R}^n$ is homeomorphic to $\boldsymbol{R}^n$ but the closed ball in $\boldsymbol{R}^n$ is not.

Compactness: A *covering* of X is a collection of open sets whose union equals X. A *subcover* of a covering is some subset of the covering which is, itself, a covering. A subset of X is *compact* if every covering of X has a subcovering with finitely many elements. It is a classic theorem that a subset of $\boldsymbol{R}^n$ is compact if and only if it is closed and bounded.

σ-Compactness X is called *σ-compact* if X is a countable union of compact subsets. For instance, any closed subset of $\boldsymbol{R}^n$ is σ-compact, but only the bounded closed subsets are compact.

7.3 Topological Manifolds

Coordinate Charts: Let M be a metric space. A *coordinate chart* in M is an open set $U \subset M$ and a homeomorphism

$$h : \boldsymbol{R}^k \to U. \tag{26}$$

We write this as (U, h). This coordinate chart is said to *contain* p if $p \in U$. Here k could depend on the point – e.g. when M is the union of a line and a plane – but we're going to be interested in the case when k is the same for all points.

Basic Definition: A *topological k-manifold* is a σ-compact metric space M such that every point of M is contained in some coordinate chart.

Examples: Here are some examples of topological manifolds.

- $\boldsymbol{R}^n$ itself.
- S^n, the n-dimensional sphere.
- The surface of any polyhedron.
- The Koch snowflake.
- The square torus - i.e. the square with sides identified.

The simplest example of a σ-compact metric space which is not a topological manifold is the union of the coordinate axes in $\boldsymbol{R}^2$.

Overlap Functions: Suppose that M is a topological manifold. Suppose that (U_1, h_1) and (U_2, h_2) are two coordinate charts in M. Suppose that these charts overlap. That is, the set $V = U_1 \cap U_2$ is nonempty. Then we have a map

$$h_2^{-1} \circ h_1 : h_1^{-1}(V) \to h_2^{-1}(V). \tag{27}$$

This map is a homeomorphism because it is the composition of homeomorphism. The function $h_2^{-1} \circ h_1$ is called an *overlap* function.

7.4 Smooth Manifolds

Compatible Charts: Let M be a topological manifold. Two coordinate charts $U_1, U_2 \in M$ are *smoothly compatible* if the overlap function defined by these charts is not just a homeomorphism, but actually smooth.

Atlases: A *smooth atlas* $\mathcal{A}$ on M is a system of coordinate charts which are all compatible with each other. We insist that every point of M is contained in at least one chart of $\mathcal{A}$. The atlas $\mathcal{A}$ is called *maximal* if there is no additional coordinate chart, not in $\mathcal{A}$, which is compatible with all the coordinate charts in $\mathcal{A}$. Zorn's Lemma guarantees that every smooth atlas on M is contained in a maximal smooth atlas.

Main Definition: A *smooth manifold* is a topological manifold equipped with a maximal smooth atlas.

Example 1: Let $F : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a smooth map and let $q \in \boldsymbol{R}^m$ be some point. We call q a *regular value*, if for every $p \in F^{-1}(q)$, the differential $dF(p)$ is surjective. In this situation, the Implicit Function Theorem gives a coordinate chart about p, and this coordinate chart is smooth in the usual sense. So, when q is a regular value, $F^{-1}(q)$ is a smooth manifold of dimension $n - m$ assuming that it is nonempty.

Example 2: Take the unit cube in $\boldsymbol{R}^n$ and identify opposite sides in the most direct possible way. Call the resulting space X. If you want to make X into a metric space, define $d(x, y)$ to be the length of the shortest path joining x to y, where these paths are allowed to go through the identified sides. You can find coordinate charts from X into $\boldsymbol{R}^n$ which are *local isometries* i.e. distance preserving when restricted to small enough open sets. (Try this for $n = 2$ first.) The overlap functions are again local isometries and hence smooth. So, the unit cube in $\boldsymbol{R}^n$ with its sides identified is naturally a smooth n-manifold. It is known as the *square n-torus*.

7.5 Maps between Smooth Manifolds

Main Definition: Suppose that M_1 and M_2 are smooth manifolds. A map $f : M_1 \to M_2$ is *smooth* if all compositions of the form

$$h_2^{-1} \circ f \circ h_1 \tag{28}$$

are smooth, where h_1 is a homeomorphism associated to a chart in M_1 and h_2 is a homeomorphism associated to a chart in M_2. What makes this a good definition is that all the overlap functions are smooth. So, to verify the smoothness of f, you don't have to examine all the uncountably many coordinate charts in the two maximal atlases. You just to verify it for some pair of sub-atlases.

Diffeomorphisms: A map $f : M_1 \to M_2$ is a *diffeomorphism* if f is a bijection and both f and f^{-1} are smooth. It is easy to verify that the composition of smooth diffeomorphisms is again a diffeomorphism. In particular, the set of diffeomorphisms from M to itself is a group! It is written $\text{Diff}(M)$.

Exercise: Here is an interesting but somewhat difficult problem. Suppose that M is any smooth manifold and $p_1, ..., p_n \in M$ are some finite set of points. Let π be some permutation of these points. Prove that there is a diffeomorphism of M which agrees with π on these points. Try it first for $\boldsymbol{R}^2$, and then for homeomorphisms of topological manifolds. Getting the map to be smooth, on a smooth manifold, is additional work.

7.6 Riemann Surfaces

The same basic framework allows you to define other kinds of structures on topological manifolds. I'll just give one example, because it is especially important.

Complex Analytic Maps: Let $U \subset \boldsymbol{C}$ be an open set. A map $f : U \to \boldsymbol{C}$ is called *complex analytic* if it is continuously differentiable, and

$$df(p) = \begin{bmatrix} A(p) & B(p) \\ -B(p) & A(p) \end{bmatrix} \tag{29}$$

for all $p \in U$. The real valued functions $A(p)$ and $B(p)$ vary continuously with p. Geometrically, $df(p)$ is a similarity. When Equation 29 is written out in terms of the matrix of partial derivatives, it is known as the *Cauchy-Riemann equations*.

Alternate Formulation: It is an amazing fact that a complex analytic map is always smooth, and equal to a convergent power series

$$f(z) = \sum_{i=0}^{\infty} c_j (z - z_0)^j, \qquad c_j \in \boldsymbol{C} \tag{30}$$

in a neighborhood of each point $z_0 \in U$. You could take this as an alternate definition of what it means for a map to be complex analytic.

Main Definition: A *Riemann Surface* is a 2-dimensional smooth manifold such that all the overlap functions defined by its atlas are complex analytic.

8 Tangent Spaces and Orientation

8.1 Smooth Curves

Let M be a smooth manifold and $p \in M$ be a point. A *curve* on M through p is a smooth map

$$\phi : (-\epsilon, \epsilon) \to M \tag{31}$$

with $\phi(0) = p$. To say that ϕ is smooth in the neighborhood of some point s is to say that $h^{-1} \circ \phi$ is smooth at s, where (h, U) is a coordinate chart and $\phi(s) \in U$. This definition does not depend on the coordinate chart, because the overlap functions are all diffeomorphisms.

Let ϕ_1 and ϕ_2 be two smooth curves through p. We write $\phi_1 \sim \phi_2$ if

$$d(h^{-1} \circ \phi_1)|_0 = d(h^{-1} \circ \phi_2)|_0.$$

Again, this is independent of the choice of coordinate chart used. The equivalence class of ϕ is denoted $[\phi]$, so we are saying that $[\phi_1] = [\phi_2]$.

We say that a *tangent vector* at $p \in M$ is an equivalence class of regular curves through p. We let $T_p(M)$ be the set of tangent vectors at p.

8.2 Vector Space Structure

We would like to show that $T_p(M)$ is a vector space, and not just a set. Suppose that M is k-dimensional, so that our coordinate charts are maps from $\boldsymbol{R}^k$ to M. Given a vector $V \in \boldsymbol{R}^k$, let L_V denote the parametrized straight line through the origin whose velocity is V. That is $L_V(t) = tV$.

Let (U, h) be a coordinate chart with $h(0) = p$. We define a map

$$dh : \boldsymbol{R}^k \to T_p(M)$$

by the rule

$$dh(V) = [h \circ L_V].$$

Lemma 8.1 *dh is injective.*

Proof: Suppose that $dh(V) = dh(W)$. Then $[h \circ L_V] = [h \circ L_W]$. But we can use the chart (h, U) to measure the equivalence. So,

$$d(h^{-1} \circ h \circ L_V)|_0 = d(h^{-1} \circ h \circ L_W)|_0.$$

But then

$$V = d(L_V)|_0 = d(L_W)|_0 = W.$$

This completes the proof. ♠

Lemma 8.2 *dh is surjective.*

Proof: Let $[\phi] \in T_p(M)$ be some tangent vector. Let V be the velocity of the curve $h^{-1} \circ \phi$. By construction $dh(V) \sim \phi$. ♠

Now we know that dh is a bijection from $\boldsymbol{R}^k$ to $T_p(M)$. We define the vector space on $T_p(M)$ in the unique way which makes h a vector space isomorphism. That is,

$$dh(V) + dh(W) = dh(V + W), \qquad r\ dh(V) = dh(rV).$$

Lemma 8.3 *The vector space structure on $T_p(M)$ is independent of the choice of coordinate chart.*

Proof: Suppose that h_1 and h_2 are two coordinate charts having the property that $h_1(0) = h_2(0) = p$. Let

$$\phi = h_2^{-1} \circ h_1$$

be the overlap function. Since ϕ is a diffeomorphism, $d\phi|_0$ is a vector space isomorphism. We just have to check that

$$d(h_2 \circ \phi) = dh_2 \circ d\phi.$$

Choose some vector $V \in \boldsymbol{R}^k$ and consider the two curves

1. $h_2 \circ \phi(L_V)$
2. $h_2 \circ L_W$, where $W = d\phi_0(V)$.

We want to show that these curves are equivalent. We can measure this equivalence using the chart (U_2, h_2). We want to see that $\phi \circ L_V$ and L_W have the same velocity at 0. The velocity of L_W at 0 is just W. The velocity of $\phi \circ L_V$ at 0 is, by definition, $d\phi(V)$. This is W. So, these two curves are equivalent. ♠

Now we know that $T_p(M)$ is a k-dimensional real vector space at each point $p \in M$.

8.3 The Tangent Map

Suppose M and N are smooth manifolds and $f : M \to N$ is a smooth map. Given $p \in M$ let $q = f(p)$. We have the differential map:

$$df|_p : T_p(M) \to T_q(N),$$

defined as follows: Given any $[\phi] \in T_p(M)$ define

$$df([\phi]) = [f \circ \phi].$$

Lemma 8.4 *This definition is independent of all choices.*

Proof: Suppose that ϕ_1 and ϕ_2 are two curves with $\phi_1 \sim \phi_2$. We want to see that $f \circ \phi_1 \sim f \circ \phi_2$. Let (U, g) be a coordinate chart for M with $p = g(0)$ and let (V, h) be a coordinate chart for N with $q = h(0)$. We are trying to show that

$$d(h^{-1} \circ f \circ \phi_1)|_0 = d(h^{-1} \circ f \circ \phi_2)|_0.$$

Note that

$$h^{-1} \circ f \circ \phi_j = (h^{-1} \circ f \circ g) \circ (g^{-1} \circ \phi_j).$$

The maps on the right hand side are maps between Euclidean spaces, and the chain rule applies. Since $\phi_1 \sim \phi_2$, we know that

$$d(g^{-1} \circ \phi_1)|_0 = d(g^{-1} \circ \phi_2)|_0,$$

because $\phi_1 \sim \phi_2$. The desired equality now follows from the chain rule. ♠

Let's check that our new definition of df gives us the same definition in cases we have already worked out.

Lemma 8.5 *If $M = R^k$ and $N = R^m$ and $f(0) = 0$, then the definition of df agrees with the usual one.*

Proof: For Euclidean spaces, we can always use the identity coordinate charts. There is a canonical isomorphism from $T_p(M)$ and $\boldsymbol{R}^k$ which maps $[\phi]$ to the velocity of ϕ at 0. Note that $df_{\rm old}$ maps V to the velocity of $f \circ \phi$. But this is just the velocity of $df_{\rm new}([\phi])$. ♠

Lemma 8.6 *If $M = \boldsymbol{R}^k$ and $f : M \to N$ is a coordinate chart, then df agrees with the initial definition of df given in terms of straight lines.*

Proof: The previous definition tells us that $df(V) = [f \circ L_V]$. But this matches the new definition, since every tangent vector in $T_p(M)$ can be represented by some L_V. ♠

Now let's talk about the Chain Rule.

Lemma 8.7 *Smooth maps between manifolds obey the chain rule.*

Proof: Suppose $f_{12} : M_1 \to M_2$ and $f_{23} : M_2 \to M_3$ are smooth maps. Then

$$d(f_{23} \circ d_{12})$$

maps the tangent vector $[\phi]$ to $[f_{23} \circ f_{12} \circ \phi]$. But this is clearly the same as $df_{23} \circ df_{12}[\phi]$. ♠

Even though we have established the chain rule, we don't yet know that df is a linear map. So, here's this final result.

Lemma 8.8 *df is a linear map.*

Proof: Let g and h be coordinates for M and N, as above. Introduce the map

$$\psi = h^{-1} \circ f \circ g.$$

Note that

$$f = h \circ \psi \circ g^{-1}.$$

By the Chain Rule, we have

$$df|_p = (dh) \circ (d\psi) \circ (dg)^{-1}.$$

Here $d\psi$ means $d\psi|_0$. All three of the maps on the right are linear maps, so df is as well. ♠

8.4 Orientations on Manifolds

Orientations on a Vector Space: Let V be a finite dimensional vector space over $\boldsymbol{R}$. Let $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ be two bases for V. We have the transition matrix T_{ij} which expresses the identity map $I : V \to V$ relative to these two bases. Call this matrix T. We call the two bases equivalent if $\det(T) > 0$. By construction, this is an equivalence relation, and there are precisely two equivalence classes. An *orientation* of V is a choice of one of the equivalance classes.

Behavior under Linear Isomorphism: If V and W are vector spaces and $T : V \to W$ is a vector space isomorphism, then T respects the equivalence relations used to define orientations. So, T maps the set of two orientations on V to the set of two orientations on W.

Pointwise Orientations: Let M be a smooth manifold and $S \subset M$ be some set. A *pointwise orientation* on S is a choice of orientation on $T_p(M)$ for each $p \in S$.

Suppose that M and N are smooth manifolds and $f : M \to N$ is a smooth and injective map. Let $S \subset M$ be some set and let $T = f(M)$. Let $p \in S$ and $q = f(p) \in T$. The differential df_p is linear, and hence induces a map from the set of (two) orientations on $T_p(M)$ to the set of (two) orientations on $T_q(N)$. So, df maps a pointwise orientation on M to a pointwise orientation on N.

Constant Orientations: When $M = \boldsymbol{R}^k$, there are two *constant orientations*. In either case, we just identify all the tangent spaces of M by translation, and take the same orientation at each point. If $U, V \subset \boldsymbol{R}^n$ is an open set and $h : U \to V$ is a diffeomorphism, then dh maps a constant orientation on U to a constant orientation on V. The point is that the determinant of dh never changes sign.

The result here is worth pondering. Even though dh could vary from point to point, on the level of orientations it is always a constant map.

Local Orientations: Let M be a manifold and let $U \subset M$ be an open set. A pointwise orientation on U is a *local orientation* if the orientation is the image of a constant orientation under a coordinate chart. It follows from the chain rule, and from the facts already mentioned about constant orientations, that this definition is independent of coordinate chart.

Global Orientations: A global orientation on M is a pointwise orientation which is a local orientation relative to every coordinate chart. If M has a global orientation, then M is said to be *orientable/*

9 Tensor Transformations

Let V and W be vector spaces and let $M : V \to W$ be a linear transformation. The map M gives a linear transformation

$$M^* : T^r(W) \to T^r(V). \tag{32}$$

Note that V and W have switched. Let $T : W^r \to \boldsymbol{R}$ be a tensor of type r. We have the tensor $M^*T : V^r \to \boldsymbol{R}$ defined by the equation

$$M^*(T)(V_1, ..., V_r) = T(M(V_1), ..., M(V_r)). \tag{33}$$

In other words, we map $V_1, ..., V_r$ into W and then apply the tensor to them. Everything involved is linear, so M^* is a linear map. The goal of these notes is to explain the action of M^*.

Let $\{v_1, ..., v_m\}$ is a basis for V and $\{w_1, ..., w_n\}$ is a basis for W. We have the formula

$$M(v_i) = \sum_{k=1}^{n} M_{ik} w_k. \tag{34}$$

The goal is to express the map M^* in terms of these coefficients.

There are three cases, the first of which is just a warm-up: the linear functional case, the general case, and the alternating case.

9.1 Linear Functional Case

We are interested in $M^* : W^* \to V^*$. We have the dual bases $\{v_1^*, ..., v_m^*\}$ and $\{w_1^*, ..., w_n^*\}$. Here $v_i^*(v_j) = 1$ if $i = j$ and 0 otherwise. Same goes for w_i^*. The matrix for M^* is just the transpose of the matrix for M.

To figure out the matrix for M^*, we just have to see that $M^*(w_j^*)$ does to v_i. We compute

$$M^*(w_j^*)(v_i) =$$

$$w_j^*(M(v_i)) =$$

$$w_j^*(\sum_{k=1^n} (M_{ik} w_k)) =$$

$$\sum_{k=1}^{n} w_j^*(M_{ik} w_k) =$$

$$M_{ij}.$$

In short,

$$M^*(w_j^*)(v_i) = M_{ij}. \tag{35}$$

But that means that

$$M^*(w_j^*) = \sum_{k=1}^{m} M_{ij} v_i^* \tag{36}$$

This is why the matrix for M^* is just the transpose of M_{ij}.

9.2 General Case

Let's introduce the multi-index notation. Let $I = (i_1, ..., i_r)$ be an r-tuple of numbers. We write

$$v_I^* = v_{i_1}^* \otimes ... \otimes v_{i_r}^*. \tag{37}$$

We write the same thing for w_I^*. Also, we write

$$v_I = (v_{i_1}, ..., v_{i_r}).$$

This is just an r-tuple of vectors. We have $v_I^*(v_J) = 1$ if $I = J$ and 0 otherwise.

We want to figure out what $M^*(w_J^*)$ does to v_I. This gives the component M_{IJ}^* of the giant matrix representing M^*.

We compute

$$M^*(w_J)(v_I) =$$

$$w_J(M(v_I)) =$$

$$w_J(M(v_{i_1}), ..., M(v_{i_r})) =$$

$$w_{j_1}^* \otimes ... w_{j_r}^*(M(v_{i_1}), ..., M(v_{i_r})) =$$

$$w_{j_1}^*(M(v_{i_1})) \times ... \times w_{j_r}^*(M(v_{i_r})) =$$

$$M_{i_1 j_1} ... M_{i_r j_r}.$$

So, the bottom line is that

$$M_{IJ} = M_{i_1, j_1} ... M_{i_r j_r}. \tag{38}$$

9.3 Alternating Case

The basis elements for $\wedge^r(V^*)$ are given by

$$[v_I^*] = A(v_I^*) = v_{i_1} \wedge ... \wedge v_{i_r}.$$

Sinilarly for $\wedge^r(W^*)$. The tensor $M^*([w_J]^*)$ is some linear combination of the various $[v_I]^*$. We want to find the coefficients. We have

$$[w_J^*] = \sum_\sigma \epsilon(\sigma) w_{\sigma J}^*. \tag{39}$$

Here σ is a permutation, and $\epsilon(\sigma)$ is the sign of σ, and σJ denotes the multi-index you get when you permute the entries of J according to the action of σ.

Now let's take I to be an increasing multi-index: $i_1 < ... < i_r$. From the previous case, and linearity, we have

$$M^*([w_J^*])(v_I) = \sum_\sigma \epsilon(\sigma) M_{I,\sigma J} = \sum_\sigma \epsilon(\sigma) M_{i_1\sigma(j_1)}, ..., M_{i_r,\sigma(j_r)}). \tag{40}$$

This last expression is just the determinant of the $r \times r$ matrix you get by taking I rows of M and the J columns.

9.4 Crucial Special Case

Suppose that $V = W$ and $r = n = \dim(V)$. Then the transformation law tells us that M^* is just multiplication by $\det(M)$. In particular, M^* is the identity map if $\det(M) = 1$.

10 Partitions of Unity

10.1 The Result

Let M be a smooth manifold. This means that

- M is a metric space.
- M is a countable union of compact subsets.
- M is locally homeomorphic to $\boldsymbol{R}^n$. These local homeomorphisms are the coordinate charts.
- M has a maximal covering by coordinate charts, such that all overlap functions are smooth.

Let $\{\Theta_\alpha\}$ be an open cover of M. The goal of these notes is to prove that M has a partition of unity subordinate to $\{\Theta_\alpha\}$. This means that there is a countable collection $\{f_i\}$ of smooth functions on M such that:

- $f_i(p) \in [0, 1]$ for all $p \in M$.
- The support of f_i is a compact subset of some Θ_α from the cover.
- For any compact subset $K \subset M$, we have $f_i = 0$ on K except for finitely many indices i.
- $\sum f_i(p) = 1$ for all $p \in M$.

The *support* of f_i is the closure of the set $p \in M$ such that $f_i(p) > 0$.

These notes will assume that you already know how to construct bump functions in $\boldsymbol{R}^n$. Note: I deliberately picked a weird letter for the cover, so that it doesn't interfere with the rest of the construction.

10.2 The Compact Case

As a warm-up, let's consider the case when M is compact. For every $p \in M$ there is some open set V_p such that

- $p \in V_p$.
- $V_p \subset \Theta_\alpha$ for some Θ_α from our cover.

- V_p is contained in a coordinate chart.

Using the fact that we are entirely inside a coordinate chart, we can construct a bump function $f : M \to [0,1]$ such that $f(p) > 0$ and the support of f is contained in a compact subset of V_p. Let $W_p \subset V_p$ denote the set of points where $f > 0$. Then W_p is an open set which contains p. Call W_p a *nice open set*.

The set $\{W_p | \; p \in M\}$ is an open covering of M. Since M is compact, we can find a finite number $W_1, ..., W_m$ of nice open sets such that $M = \bigcup W_i$. Let $g_1, ..., g_m$ be the functions associated to these open sets. By construction, $g_i > 0$ on W_i. This means that the sum $\sum g_i$ is positive on M. Define

$$f_i = \frac{g_i}{\sum g_i}. \tag{41}$$

Then $f_1, ..., f_m$ make the desired partition of unity.

The rest of the notes deal with the case when M is not compact.

10.3 Fattening Compact Sets

We need two technical lemmas.

Lemma 10.1 *Let $p \in M$ be any point. For all sufficiently small ϵ, the ball of radius ϵ has compact closure in M.*

Proof: There is some neighborhood U of p which is homeomorphic to $\boldsymbol{R}^n$. Let $\phi : U \to \boldsymbol{R}^n$ be a homeomorphism. Choose some closed ball $B \subset \boldsymbol{R}^n$ which contains $\phi(p)$. Consider $\phi^{-1}(B)$. This is a compact subset of M, and it contains the open set $U' = \phi^{-1}(\text{interior}(B))$. Any sufficiently small open ϵ ball Δ about p will be contained in U' and hence will have closure contained in the compact set $\phi^{-1}(B)$. A closed subset of a compact set is compact. Hence, the closure of Δ is compact. This is what we wanted to prove. ♠

Lemma 10.2 *If $X \subset M$ is compact, then there exists some compact subset Y such that X is contained in the interior of Y.*

Proof: For each $p \in X$, there is some ϵ ball Δ_p whose closure in M is compact. The union of such balls covers X. Since X is compact, we can take a finite subcover. That is, $X \subset \Delta_1 \cup ... \cup \Delta_m$. Let Y be the union of the closures of these balls. Since Y is a finite union of compact sets, Y is compact. The interior of Y contains the union of these open balls, and hence contains X. ♠

10.4 Cleaning up the Compact Sets

Lemma 10.3 *There exists a countable collection $\{K_i\}$ of compact sets such that K_i is contained in the interior of K_{i+1} for all i, and $M = \bigcup K_i$.*

Proof: We know already that $M = \bigcup K_i$, where K_i is compact and the union is countable. Replacing K_m by $K_1 \cup ... \cup K_m$, it suffices to consider the case when $K_1 \subset K_2 \subset K_3 ....$

Suppose we know already that K_i is contained in the interior of K_{i+1} for $i = 0, ..., m$. By the preceding lemma, we can replace K_{m+2} by a larger compact set L_{m+2} which contains K_{m+2} in its interior. Now we redefine $K_{m+3} = L_{m+2} \cup K_{m+3}$ and $K_{m+4} = L_{m+2} \cup K_{m+3} \cup K_{m+4}$, etc. The new collection of compact sets has $K_i \subset K_{i+1}$ for all $i = 0, ..., m + 1$. By induction, we can get this property for all i. ♠

Lemma 10.4 *We can write $M = \bigcup L_i$, where L_i is compact for all i, and $L_i \cap L_j = \emptyset$ if $j < i - 1$.*

Proof: We know that $M = \bigcup K_i$, where each K_i is compact, and K_i is contained in the interior of K_{i+1} for all i. Define

$$L_i = K_i - \text{interior}(K_{i-1}). \qquad (42)$$

Note that L_i is disjoint from K_j for $j < i - 1$. Hence L_i is disjoint from L_j for $j < i - 1$. By construction L_i is a compact set minus an open set. In other words, L_i is the intersection of a compact set and a closed set. Hence L_i is compact. Also, $M = \bigcup L_i$. ♠

10.5 The Main Construction

We keep the notation from the previous section. Consider L_i. Each $p \in L_i$ has an open metric ball U such that

- U is disjoint from L_j for all $j < i - 1$. This uses the fact that there is a minimum positive distance between U_i and U_j for all $j < i - 1$.

- U is contained in some Θ_α from our cover.

- U is contained in a coordinate chart.

As in the compact case, we can construct a bump function f such that $f(p) > 0$ and the support of f is contained in a compact subset of U. Let $W \subset U$ denote the set where $f > 0$. Call W a nice set. Since L_i is compact, we can cover L_i by finitely many nice sets, say $W_{i1}, ..., W_{im_i}$. (The number depends on i.)

Now we consider the covering

$$W_{11}, ..., W_{1i_1}, W_{21}, ..., W_{2i_2}, ...$$

We rename these sets $X_1, X_2, X_3, ...$ and let g_1, g_2, g_3 be the associated functions. These functions have the following properties.

- For every $p \in M$, there is some g_i such that $g_i > 0$. This comes from the fact that $p \in L_j$ for some j, and then p is contained in some nice set on our list.

- Any compact set only intersects finitely many X_i. The point is that any compact set is contained in the union of finitely many L_i.

- The support of each g_i is contained in some Θ_α from the original cover. This comes from the fact that the support of g_i is the closure of a nice set.

Now we define $f_i = g_i / \sum g_j$, as in the compact case. The sum is locally finite at each point. This gives us the partition of unity.

11 The Poincare Lemma

The purpose of these notes is to explain the proof of Poincare's lemma from the book in somewhat less compressed form.

11.1 De Rham Cohomology

Let $U \subset \boldsymbol{R}^n$ be any open set. Recall that $\Omega^r(U)$ is the space of smooth r-forms on U. On $\Omega^r(U)$ we have the basic equation $d^2 = 0$. We let

$$Z^r(U) \subset \Omega^r(U)$$

denote the set of forms ω such that $d\omega = 0$. We let

$$B^r(U) \subset \Omega^r(U)$$

denote the set of forms ω such that $\omega = d\alpha$ for some $\alpha \in \Omega^{r+1}(U)$. Both $Z^r(U)$ and $B^r(U)$ are vector spaces, and $B^r(U) \subset Z^r(U)$. The space $Z^r(U)$ is often called the set of *closed forms* on U and the set $B^r(U)$ is often called the set of *exact forms* on U.

We define

$$H^r(U) = \frac{Z^r(U)}{B^r(U)}. \tag{43}$$

Here we are taking the quotient of vector spaces. The vector space $H^r(U)$ is often called the r-th *de Rham cohomology* of U.

The de Rham Cohomology is a diffeomorphism invariant. Suppose that $f : U \to V$ is a diffeomorphism. Then $f^*d = df^*$. This means that the pullback f^* maps $Z^r(V)$ into $Z^r(U)$ and $B^r(V)$ into $B^r(U)$. So, f induces a map $f^* : H^r(V) \to H^rU)$. Since f^{-1} is also smooth we see that f^* has an inverse, namely $(f^{-1})^*$. Hence f^* is an isomorphism. In other words, if U and V are diffeomorphic then $H^r(U)$ and $H^r(V)$ are isomorphic.

11.2 The Main Result

A domain $U \subset \boldsymbol{R}^n$ is *star shaped with respect to* $p \in \boldsymbol{R}^n$ if, for each $q \in U$, the entire segment $\overline{pq}$ lies in U. We say that U is star-shaped if U is star-shaped with respect to some point. Here is the main result.

Lemma 11.1 (Poincare) *If $U \subset \boldsymbol{R}^n$ is open and star-shaped then we have $H^r(U) = 0$ for all $r \geq 0$. In other words, every closed form on U is exact.*

First I will prove an algebraic result that works for any open subset of $\boldsymbol{R}^n$, star shaped or not, and then I'll apply the result to the Poincare Lemma.

11.3 An Algebraic Result

Let $U \subset \boldsymbol{R}^n$ be any open set, not necessarily star-shaped. Let $\widehat{U}$ be an open subset of $\boldsymbol{R}^{n+1}$ which contains $U \times [0,1]$. We set $x_{n+1} = t$.

We define $g_j : U \to U \times \{j\} \subset \widehat{U}$ by the formula

$$g_j(p) = (p, j). \tag{44}$$

here $j = 0$ and $j = 1$. Let $g_j^* : \Omega^r(\widehat{U}) \to \Omega^r(U)$ be the pull-back operator.

We now construct a map $J : \Omega^{s+1}(\widehat{U}) \to \Omega^s(U)$ for every value of $s \geq 0$. The map will have the property that

$$(-1)^r \Big(dJ - Jd \Big) = g_0^* - g_1^*. \tag{45}$$

In the context of de Rham cohomology, the map J is called a *chain homotopy*.

To define J we just have to specify what it does to a form $\omega = f dx_I$ and then extend linearly. Here $I = (i_1, ..., i_{r+1})$ is an increasing multi-index. If $i_{r+1} < n+1$ we define $J(\omega) = 0$. If $i_{r+1} = n+1$ we define

$$J(\omega) = F dx_{I'}, \qquad F(p) = \int_0^1 f(p,t)\ dt, \qquad I' = (i_1, ..., i_r). \tag{46}$$

Here $p = (x_1, ..., x_n)$. You might say that J "integrates out" the last coordinate. Since both sides of Equation 45 are linear, it suffices to check Equation 45 on our form ω.

Case 1: Suppose that $i_{r+1} < n+1$. We note first that

$$g_j^*(\omega)|_p = f(p,j) dX_I.$$

All we are doing is restricting ω to the slice $U \times \{j\}$. We already know that $J(\omega) = 0$. Hence $dJ(\omega) = 0$ as well. We have

$$d\omega = (-1)^r \frac{\partial f}{\partial t} dX_I \wedge dt + \text{other terms}.$$

The other terms do not involve dt. This means that $Jd(\omega)$ only involves the first term. By the Fundamental Theorem of Calculus,

$$\int_0^1 \frac{\partial f}{\partial t}(p,t) = f(p,1) - f(p,0).$$

Therefore,

$$Jd(\omega)|_p = (-1)^r(f(p,1) - f(p,0))dX_I = (-1)^r(g_1^*(\omega) - g_0^*(\omega)).$$

Combining this equation with the fact that $dJ(\omega) = 0$ gives us Equation 45.

Case 2: Suppose that $i_{r+1} = n+1$. Thus $\omega = f dX_{I'} \wedge dt$. Note that the image of $Dg_j(T_p(U))$ is perpendicular to the t-direction. For this reason $g_j^*(\omega) = 0$. So, in this case we just need to establish that $dJ(\omega) = Jd(\omega)$.

We have

$$dJ(\omega) = d(F \wedge dx_{I'}) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_j} dx_i \wedge dx_{I'}.$$

We do not have a term in the $n+1$-th coordinate because we would get $dt \wedge dt = 0$ in this case. At the same time

$$d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \wedge dx_j \wedge dx_{I'} \wedge dt. \tag{47}$$

Differentiating under the integral sign, we have

$$\int_0^1 \frac{\partial f}{\partial x_i}(p,t)\ dt = \frac{\partial F}{\partial x_j}(p). \tag{48}$$

Combining Equations 47 and 48 with the definition of J, we see that

$$Jd(\omega) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_j} dx_i \wedge dx_{I'} = dJ(\omega).$$

11.4 The Application

Now we prove the Poincare Lemma as an application of Equation 45. We return to the case when U is a star-shaped domain. By symmetry, it suffices to consider the case when U is star-shaped with respect to 0. Let $\widehat{U} \subset \boldsymbol{R}^{n+1}$ be the set of all pairs (p,t) such that $tp \in U$. Here we are just scaling p to t.

Since U is open, the set $\widehat{U}$ is open. Since U is star-shaped with respect to 0, the set $\widehat{U}$ contains $U \times [0,1]$.

We introduce a map $\Phi : \widehat{U} \to U$ given by

$$\Phi(p,t) = tp. \tag{49}$$

Let g_0 and g_1 be the maps from the previous subsection. The composition $g_0 \circ \Phi$ is the 0-map and the composition $g_1 \circ \Phi$ is the identity map. This gives us

$$g_0^*\Phi^*(\omega) = 0, \qquad g_1^*\Phi^*(\omega) = \omega \tag{50}$$

for any $\omega \in \Omega^r(U)$.

Suppose that $\omega \in \Omega^r(U)$ satisfies $d\omega = 0$. We define

$$\alpha = (-1)^r J\Phi^*(\omega). \tag{51}$$

Note that

$$Jd\Phi^*(\omega) = J\Phi^*(d\omega) = 0.$$

Therefore

$$d\alpha = (-1)^r dJ\Phi^*(\omega) =$$

$$(-1)^r \Big(dJ\Phi^*(\omega) - Jd\Phi^*(\omega) \Big) =$$

$$g_1^*\Phi^*(\omega) - g_0^*\Phi^*(\omega) = \omega - 0 = \omega.$$

This proves the Poincare Lemma.

12 The Brouwer Fixed Point Theorem

The purpose of these notes is to explain the proof of Brouwer's fixed point theorem using differential forms. This proof is similar to what is in Browder's book, but it emphasizes different points.

12.1 No Retraction Theorem

Let M be a smooth compact oriented n-manifold-with-boundary. Let ∂M be the boundary, oriented so that Stokes' Theorem is true for M. Just for fun, I will prove the no retraction theorem in more generality than I did in class. Say that a map $g : \partial M \to \partial M$ is *nice* if there are (relatively) open subsets $U, V \subset \partial M$ such that $g : U \to V$ is a diffeomorphism and $g^{-1}(V) = U$.

Theorem 12.1 *There is no smooth map $f : M \to \partial M$ such that the restriction of f to ∂M is nice.*

Proof: We suppose f exists and derive a contradiction. Using a bump-function construction, we can choose a smooth $(n-1)$-form α on M which is supported in U such that $\int_{\partial M} \alpha = 1$. Let $\beta = f^*(\alpha)$. We have

$$d\beta = df^*(\alpha) = f^*(d\alpha) = f^*(0) = 0.$$

Hence $\int_M d\beta = 0$. By Stokes' Theorem, $\int_{\partial M} \beta = 0$. But, by the change of variables formula for diffeomorphisms, and the properties of f, we have $\int_{\partial M} \beta = \int_{\partial M} \alpha = 1$. This is a contradiction. ♠

Since the identity map is nice, we see that there is no smooth map $f : M \to \partial M$ which restricts to the identity on ∂M.

12.2 Brouwer's Theorem for Smooth Maps

Let B^n be the n-ball.

Theorem 12.2 *If $g : B^n \to B^n$ is a smooth map then Then $g(p) = p$ for some $p \in B^n$.*

Proof: We suppose that this is false and derive a contradiction. If it never happens that $g(p) = p$ then we can define $f(p) \in \partial B_n$ to be the point where the ray from $f(p)$ through p intersects ∂B_n. This map is smooth on the interior of B^n because you find the point $f(p)$ by solving a quadratic equation whose coefficients vary smoothly with p.

The map $f : B^n \to \partial B^n$ seems to violate the no-retraction theorem. However, we have not really shown that f is smooth at points on ∂B^n. The issue is that the extension idea does not necessarily work for points outside B_n. The ray from $f(p)$ to p might not hit ∂B_n at all. (In the book, Browder does not worry about this.) Here is a trick to deal with this.

Define a new function $h : \boldsymbol{R}^n - \{0\} \to \partial B^n$ to be radial projection. Also, choose some emall $\epsilon > 0$ and define a bump function β which is 1 on the ball of radius $1 - 2\epsilon$ centered at the origin and 0 outside the ball of radius $1 - \epsilon$. Consider the new function

$$\phi(p) = h(\beta(p)f(p) + (1 - \beta(p))h(p)).$$

This is a smooth function which maps B^n into ∂B^n provided that ϵ is small enough so that the chord connecting $f(p)$ and $h(p)$ does not contain the origin when $\|p\| \geq 1 - 2\epsilon$.

By construction $\phi = h$ on ∂B_n. Hence the restriction of ϕ to ∂B_n is the identity. Using ϕ (rather than f) we get a map which is smooth on all of B_n. This contradicts the no retraction theorem. ♠

12.3 Convolution

Our final goal is to prove Brouwer's Theorem for continuous maps. As a prelude, we explain how to approximate a bounded continuous function, defined on a bounded open subset U of $\boldsymbol{R}^n$ by a smooth function.

Let $f : U \to \boldsymbol{R}$ be a continuous bounded function. Let g be any smooth function. We define

$$f * g(x) = \int_U f(y)g(x - y)\ dy. \tag{52}$$

Here $x, y \in \boldsymbol{R}^n$.

Given some standard basis vector e_i we compute

$$\frac{f * g(x + te_i) - g^*(x)}{t} = \int_U f(y)\frac{g(x + te_i - y) - g(x - y)}{t}\ dy =$$

$$\int_U f(y)\frac{\partial g}{\partial x_i}(x-y)\ dy + \int_U f(y)E_t(x-y)\ dy$$

Here E_t is an "error function" which uniformly tends to 0 on U as $t \to 0$. So, taking a limit, we see that the ith partial derivative of $f * g$ exists and

$$\frac{\partial(f*g)}{\partial x_i} = f * \frac{\partial g}{\partial x_i}. \tag{53}$$

Iterating this result and using the smoothness of g we see that $f * g$ has partial derivatives of all orders and therefore is smooth.

For any N we can choose g_N so that $\int_{\boldsymbol{R}^n} g_N = 1$ and so that the support of g_N is contained in the $1/N$ neighborhood of the origin. In this case $f * g_N$ is smooth and quite close to f. More precisely $f * g_N$ converges uniformly to f on U as $N \to \infty$.

12.4 Brouwer's Theorem for Continuous Maps

Suppose now that $g : B^n \to B^n$ is a continuous map. We first extend g so that it is continuous and defined an an open neighborhood U of B_n. Applying the convolution trick to each coordinate function of g we produce a sequence $\{g_N\}$ of smooth maps from B_n to B_n which converge uniformly to g.

By the smooth version of Brouwer's Theorem there are points $p_N \in B^n$ such that $g_N(p_N) = p_N$. Let p be any accumulation point of $\{p_N\}$. If $g(p) \neq p$ then, by continuity, $g_N(p_N) \neq p_N$ for N sufficiently large. Hence $g(p) = p$. This proves Brouwer's fixed point theorem for continuous maps.

13 Integrating Functions on Manifolds

These notes deal with integrating functions on (Riemannian) manifolds. We already know how to integrate k-forms on a k-manifold but the topic here is how to deal with functions. The purpose of these notes is to clarify what is going on by explaining things in terms of abstract manifolds.

The general way it works is that one can integrate functions on a *Riemannian manifold*, because the Riemannian metric defines a canonical volume form locally. The canonical form is defined everywhere, up to a sign. The sign can't work out globally if the manifold is non-orientable, but there is a trick using partitions of unity to make use of these local volume forms even in the non-orientable case.

When one has a submanifold in $\boldsymbol{R}^n$, there is a canonical Riemannian metric which just comes from the restriction of the dot product. So, you can use the abstract theory to integrate functions submanifolds of $\boldsymbol{R}^n$. The final theory turns out to be equivalent to what is done in the book.

13.1 Inner Products and Volume Forms

Let V be a finite dimensional real vector space. An *inner product* on V is a map $Q : V \times V \to \boldsymbol{R}$ such that

1. Q is a symmetric 2-tensor.
2. $Q(w, w) > 0$ for all $w \neq 0$.

Lemma 13.1 *There exists an orthonormal basis for Q.*

Proof: Given a basis $\{v_1, ..., v_n\}$ for V we can perform the usual Gram-Schmidt process for creating an orthonormal basis with respect to Q. The procedure works like this.

- Replace v_1 by
$$w_1 = v_1/\sqrt{Q(v_1, v_1)}$$
so arrange that $Q(w_1, w_1) = 1$.
- Assuming that $w_1, ..., w_k$ have been constructed, let
$$w'_{k+1} = v_{k+1} - \sum_{i=1}^{k} Q(v_{k+1}, w_i)w_i.$$

This guarantees that $Q(w'_{k+1}, w_i) = 0$ for all $i = 1, ..., k$.

- Replace w'_{k+1} by

$$w_{k+1} = w'_{k+1}/\sqrt{Q(w'_{k+1}, w'_{k+1})}.$$

This produces $w_1, ..., w_n$ such that $Q(w_i, w_j) = 1$ if $i = j$ and 0 otherwise. ♠

Remark: Notice that each w_j varies smoothly as a function of $v_1, ..., v_n$. That is, we can think of w_j as a function from V^n to V, and it is a smooth function.

Lemma 13.2 *Assume that $\boldsymbol{R}^n$ is equipped with the dot product. There is a linear transformation $T : \boldsymbol{R}^n \to V$ which is an isometry between $\boldsymbol{R}^n$ and V.*

Proof: Let $e_1, ..., e_n$ be the standard basis for $\boldsymbol{R}^n$ and let $w_1, ..., w_n$ be an orthonormal basis for V. The map $T(e_j)$ does the trick. ♠

Definition: The *adapted volume forms* on V are the two forms

$$\pm(T^{-1})^*(dx_1 \wedge ... \wedge dx_n).$$

If V also has an orientation, we can "prefer" one of these over the other.

13.2 Riemannian Manifolds

A *Riemannian metric* on a smooth manifold M is a smoothly varying choice of inner product Q_p on each tangent space $T_p(M)$. The smoothness has the following explanation. If $\alpha : \boldsymbol{R}^n \to M$ is any smooth coordinate chart, then the pullback inner product $\alpha^*(Q)$ is given by a symmetric matrix at each point of $\boldsymbol{R}^n$. We want the entries of this matrix to be smooth functions. This is the usual way we talk about smooth tensor fields on manifolds.

Suppose that M has a Riemannian metric Q. For each $p \in M$ there are two adapted volume forms associated to Q_p, and they differ only by sign. Call these two volume forms $\pm\omega_p$. Let V be a coordinate patch in M. Note that V has one of two local orientations, regardless of whether or not M is orientable. We say that the assignment $p \to \omega_p$ is *continuous* if ω_p defines the same orientation at each $p \in V$. In other words, ω_p is either always positive or always negative when evaluated on a positively oriented basis, as p varies throughout V. Notice that there are exactly 2 continuous adapted volume forms on each coordinate chart.

If M is orientable, we can make a consistent choice of a continuous adapted volume form on M. Otherwise, we have to be content with a system of continuous adapted volume forms, one per coordinate chart.

13.3 Integration of Functions

Let's continue with the same notation. Suppose that $V \subset M$ is a coordinate chart. Suppose that $f : M \to \boldsymbol{R}$ is a non-negative Borel measurable function whose support is contained in V.

We choose an orientation on V, as well as the corresponding adapted volume form ω. We then define

$$\int_M f = \int_M f\omega.$$

Notice that this is a non-negative number, and strictly positive if $f > 0$ on some open set. Were we to pick the opposite orientation, we would be integrating $-f\omega$ with respect to an oppositely oriented coordinate chart, and we would get the same answer. So, the integral is completely well defined.

Now suppose that $f : M \to \boldsymbol{R}$ is any non-negative Borel function whose support is compact. (This is automatic if M is a compact manifold.) We choose a partition of unity $\{\phi_i\}$ subordinate to some open cover by coordinate charts, and we define

$$\int_M f = \sum \int_M \phi_i f.$$

The compactness guarantees that this is just a finite sum. The same argument as for the integration of forms shows that this definition is independent of the choice of partition of unity.

Remark: If you don't like working with Borel measurable functions, you can restrict your attention to continuous functions. This is all we really need for applications in the book. For continuous functions, the integrals involved can be done by the usual Riemann integral.

Suppose now that $f : \boldsymbol{M} \to \boldsymbol{R}$ is a compactly supported function We can write $f = f_+ - f_-$, where $f_+ = \max(f, 0)$, and $f_- = f - f_+$. Then we define

$$\int_M f = \int_M f_+ - \int_M f_-.$$

13.4 Euclidean Submanifolds

Suppose now that M is an n-dimensional submanifold of $\boldsymbol{R}^N$. There is a canonical Riemannian metric on M, namely

$$Q_p(V, W) = V \cdot W, \qquad \forall V, W \in T_p(M).$$

We then integrate functions on M with respect to the system of volume forms adapted to M on coordinate charts.

It is worth pointing out why these volume forms are smooth. Let $V \subset M$ be a coordinate patch on M and let $\alpha : \boldsymbol{R}^n \to V$ be a coordinate map. We can get a basis at each point $p \in V$ using $\alpha_*(e_1), ..., \alpha_*(e_n)$. This basis varies smoothly. We can then perform Gram-Schmidt to get a smoothly varying orthonormal basis. The matrix entries of the adapted quadratic form are rational-function entries of the coefficients of the orthonormal bases, to they vary smoothly as well.

13.5 Reconciling with the Book

Suppose that $f : M \to \boldsymbol{R}$ is a positive function whose support is contained in the coordinate patch V. Let α be a coordinate chart whose image is V. Then the expression

$$\sqrt{\det(A^t A)}, \qquad A = D\alpha$$

computes the infinitesimal volume multiplier under the action of α. That is, in each tangent space, the differential map A multiplies volume by $\det(A^t A)$, as explained in the book.

But that means that

$$\alpha^*(\omega) = \sqrt{\det(A^t A)} dx_1 \wedge ... \wedge dx_n.$$

Hence

$$\alpha^*(f\omega) = f\sqrt{\det(A^t A)} dx_1 \wedge ... \wedge dx_n.$$

So,

$$\int_M f = \int_M f\omega = \int_{\boldsymbol{R}^n} \alpha^*(f\omega) = \int_{\boldsymbol{R}^n} f\sqrt{\det(A^tA)}\ dx_1...dx_n.$$

This last expression is what is in the book.

13.6 Another Perspective

Suppose specifically that M is a hypersurface in $\boldsymbol{R}^n$. Let ν denote a unit normal field along M. Let ι_ν denote the contraction operator. Let

$$\omega = dx_1 \wedge ... \wedge dx_n$$

be the standard volume form on $\boldsymbol{R}^n$. Then $\iota_\nu(\omega)$ is the volume form along M. Here

$$\iota_\nu(V_1, ..., V_{n-1}) := \omega(\nu, V_1, .., V_n).$$

The point here is that when $V_1, ..., V_{n-1}$ is an orthonormal basis at a tangent space of M then $\nu, V_1, ..., V_{n-1}$ is an orthonormal basis for $\boldsymbol{R}^n$. In this case $\iota_\nu\omega(V_1, ..., V_{n-1}) = \pm 1$, and the sign choice varies continuously.

14 Harmonic Functions and Hodge Star

The purpose of these notes is to cover the Hodge star operator, the divergence form of Stokes' Theorem, and foundational results about harmonic functions. These notes also have 6 HW exercise in them.

14.1 The Hodge Star Operator

We'll start out by defining the Hodge star operator as a map from $\wedge^k(\boldsymbol{R}^n)$ to $\wedge^{n-k}(\boldsymbol{R}^n)$. Here $\wedge^k(\boldsymbol{R}^n)$ denotes the vector space of alternating k-tensors on $\boldsymbol{R}^n$.

Let $I = (i_1, ..., i_k)$ be some increasing multi-index of length k. That is $i_1 < i_2 < i_3 < ....$ Let $J = (j_1, ..., j_{n-k})$ be the complementary increasing multi-intex. For instance, if $n = 7$ and $I = (1, 3, 5)$ then $J = (2, 4, 6, 7)$. Let K_0 denote the full multi-index $(1, ..., n)$.

We first define $*$ on the usual basis elements:

$$*(dx_I) = \pm dx_J, \tag{54}$$

where the sign is chosen so that

$$dx_I \wedge *(dx_I) = dx_1 \wedge ... \wedge dx_n. \tag{55}$$

We often write $*dx_I$ in place of $*(dx_I)$. In general, we define

$$*\Big(\sum a_I \ dx_I\Big) = \sum a_I \ (*dx_I). \tag{56}$$

Exercise 1: For any $\omega \in \wedge^k(\boldsymbol{R}^n)$ prove that $* * \omega = (-1)^{k(n-k)}\omega$. Hint: Show this on a basis and use the anti/commutative properties of the wedge.

14.2 Rotational Symmetry

Let $O(n)$ denote the set of orthogonal transformations of $\boldsymbol{R}^n$. These are the transformations which preserve the dot-product. Note that we have $\det(M) = \pm 1$ for $M \in O(n)$. The subgroup $SO(n)$ consists of those matrices having determinant equal to 1. Our next goal is to prove that

$$M^*(*\omega) = \det(M)(*(M^*(\omega)), \qquad \forall M \in O(n). \tag{57}$$

In particular,

$$M^*(*\omega) = *(M^*(\omega)), \qquad \forall M \in SO(n). \tag{58}$$

These are meant to hold for all $\omega \in \wedge^k(\boldsymbol{R}^n)$ regardless of the values of k and n. Equation 58 is the main equation we are interested in, but for the proof it is useful to sometimes consider maps in $O(n)$ that are not in $SO(n)$.

Exercise 2: Fix some $k \in \{1, ..., n-1\}$ and let M be the map such that $M(e_k) = e_{k+1}$ and $M(e_{k+1}) = e_k$ and otherwise $M(e_j) = e_j$. So, in other words, M just swaps two of the coordinates. Prove Equation 57 for M. Hint: check the equation on a basis. Also, conclude that Equation 57 holds for any permutation matrix.

Lemma 14.1 *Let M be the element of $SO(n)$ which has the following action:*

- $M(e_j) = e_j$ *for* $j = 3, 4, 5, ...$
- $M(e_1) = e_1 \cos(\theta) + e_2 \sin(\theta)$,
- $M(e_2) = -e_1 \sin(\theta) + e_2 \cos(\theta)$.

In other words, M rotates by θ in the e_1, e_2 plane and fixes the perpendicular directions. The Equation 57 holds for M.

Proof: It suffices to check this on a basis. Consider dx_I. Let J be the complementary index. We will prove our result when $*dx_I = dx_J$ (rather than $-dx_J$.) The other case has the same kind of treatment.

Suppose first that I contains neither 1 nor 2. Then $M^*(dx_I) = dx_I$. Also, the complementary multi-index J contains both 1 and 2. Using the transformation law for forms, we have

$$M^*(dx_J) = dx_J. \tag{59}$$

We compute

$$M^*(*dx_I) = M^*(dx_J) = dx_J = *dx_I.$$

The case when I contains both 1 and 2 has the same proof.

Suppose that I contains 1 but not 2. Then $dx_I = dx_1 \wedge dx_{I'}$. Here I' is obtained from I by omitting 1. Similiarly, we have the equations

$$*(dx_1 \wedge dx_{I'}) = *dx_I = dx_J = dx_2 \wedge dx_{J'}, \qquad *(dx_2 \wedge dx_{I'}) = -dx_1 \wedge dx_{J'}.$$

Here J' is obtained from J by omitting 2. The sign change in the second calculation comes from the fact that

$$dx_2 \wedge dx_{I'} \wedge (-dx_1) \wedge dx_{J'} = dx_1 \wedge dx_{I'} \wedge dx_2 \wedge dx_{J'} = dx_I \wedge dx_J = dx_1 \wedge ... \wedge dx_n.$$

We set $C = \cos(\theta)$ and $S = \sin(\theta)$. An easy computation shows that

$$M^*(dx_1) = Cdx_1 - Sdx_2, \qquad M^*(dx_2) = Sdx_1 + Cdx_2.$$

These calculations tell us that

$$*M^*(dx_I) = *\Big((Cdx_1 - Sdx_2) \wedge dx_{I'}\Big) =$$

$$*\Big(Cdx_1 \wedge dx_{I'}\Big) - *\Big(Sdx_2 \wedge dx_{I'}\Big) = Cdx_2 \wedge dx_{J'} + Sdx_1 \wedge dx_{J'}.$$

Similarly

$$M^*(*dx_I) = M^*(dx_2 \wedge dx_{J'}) =$$

$$(Sdx_1 + Cdx_2) \wedge dx_{J'} = Sdx_1 \wedge dx_{J'} + Cdx_2 \wedge dx_{J'}.$$

The two expressions agree.

There is one more case, when I contains 2 but not 1. This case is similar to the last case, and actually follows from the last case and Exercise 1. ♠

Exercise 3: Verify Equation 59. Hint: Use basic linearity properties of the wedge product.

Exercise 4: Let $G \subset O(n)$ denote the subgroup generated by the permutation matrices and the elements M (for all θ) considered in the previous lemma. Prove that $G = O(n)$. (Hint: Starting with an arbitrarily element $T \in O(n)$ try to find an element $g \in G$ such that $g \circ T$ fixes e_1 and hence $e_1^\perp$. Then use induction on on n.) Deduce Equation 57 from this Exercise, and Exercise 2, and the previous lemma.

Now we deduce an important consequence of Equation 58. Suppose that $w_1, ..., w_n$ is any orthonormal basis of $\boldsymbol{R}^n$. We assume that this is a positively oriented basis so that there is some $M \in SO(n)$ such that $M(w_j) = e_j$ for $j = 1, ..., n$. This means that $M^*(dx_I) = dw_I$. We compute

$$*dw_I = *M^*(dx_I) = M^*(*dx_I) = M^*(\pm dx_J) = \pm dw_J.$$

The sign works out whether or not $*dx_I = dx_J$ or $*dx_I = -dx_J$. What we are saying here is that we can define $*$ with respect to any positively oriented orthonormal basis and we get the same answer as when we use the standard basis. This beautiful symmetry will help us in the next section.

14.3 Divergence Form of Stokes' Theorem

Now suppose that M is the unit ball in $\boldsymbol{R}^n$ and $S^{n-1} = \partial M$ is its boundary.

Let V be a vector field on M, say $V = (V_1, ..., V_n)$. We have the usual associated 1-form $\omega = \sum V_i dx_i$. Note that $*\omega$ is an $(n-1)$ form on M and $d(*\omega)$ is an n-form on M. Stokes' theorem, applied to $*\omega$, tells us that

$$\int_M d(*\omega) = \int_{\partial M} *\omega. \tag{60}$$

We're going to re-interpret each half of this equation.

The Left Side: A direct calculation shows that

$$d(*\omega) = \sum \partial V_i/\partial x_i \; dx_1 \wedge ... \wedge dx_n = \text{div}(V) \; dx_1 \wedge ... \wedge dx_n.$$

So, the left hand side of Equation 60 equals

$$\int_M \text{div}(V) \; dx_1...dx_n,$$

the usual integral of the divergence of a vector field.

The Right Side: Now let's consider the right hand side of Equation 60. Consider the form $*\omega$ at a point p of ∂M. We can find an oriented orthonormal basis for $\boldsymbol{R}^n$ at p, say $w_1, ..., w_n$, so that

- $w_1, ..., w_{n-1}$ is an oriented orthonormal basis for $T_p(\partial M)$.
- $\nu = (-1)^{n-1} w_n$ is the normal vector that is compatible with Stokes' theorem.

Since ω is a 1-form, we can write in our corresponding basis of 1 forms (where $dw_i = w_i^*$.)

$$\omega = \sum b_i dw_i.$$

Note that the restriction of $*dw_i$ to ∂M at p is 0 unless $i = n$. Therefore the restriction of $*\omega$ to ∂M at p equals $b_n(*dw_n)$. That is

$$*\omega|_{\partial M} = b_n(*dw_n).$$

But

$$b_n = \omega(w_n) = (-1)^{n-1}\omega(\nu) = (-1)^{n-1}V \cdot \nu.$$

Finally,

$$*dw_n = (-1)^{n-1} dw_1 \wedge ... \wedge dw_{n-1}.$$

Putting these three equations together, we get

$$*\omega|_{\partial M} = V \cdot \nu \; dw_1 \wedge ... \wedge dw_{n-1}.$$

Our theory of integrating functions on manifolds tells us that the right hand side of Equation 60 is

$$\int_{\partial M} V \cdot \nu.$$

The Interpretation: Putting everything together, we have

$$\int_M \text{div}(V) = \int_{\partial M} V \cdot \nu. \tag{61}$$

On the left hand side, we are integrating with the usual volume measure on Euclidean space, and on the right hand side we are integrating a function on an oriented manifold according to the theory explained in the class. This is a classical n-dimensional generalization of Gauss's law of electrostatics.

Really, there is nothing special we used about spheres. This result also holds when M is any compact n-dimensional manifold in $\boldsymbol{R}^n$. In particular, this result holds when M is a region bounded by two concentric spheres. This is the case of interest to us.

14.4 Green's Identity

Our argument works for any compact n-dimensional manifold $M \subset \boldsymbol{R}^n$. The point is that all we need here is for the divergence form of Stokes' Theorem to work.

Let ∇f stand for the gradient of a function f and let δf stand for the Laplacian of f. That is

$$\nabla f = \Big(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\Big), \qquad \Delta f = \sum \frac{\partial^2 f}{\partial^2 x_i}. \tag{62}$$

To say that f is *harmonic* is to say that $\delta f = 0$.

Let f and g be two smooth functions in M. Then

$$\int_{\partial M} (f\nabla g \cdot v - g\nabla f \cdot \nu) = \int_M (f\Delta g - g\Delta f). \tag{63}$$

Here is the derivation. Consider vector fields $V_1 = g\nabla f$ and $V_2 = f\nabla g$. We compute

$$\mathrm{div}(V_1) = \sum \frac{\partial}{\partial x_i}\Big(g\frac{\partial f}{\partial x_i}\Big) = \sum \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_i} + \sum g\frac{\partial^2 f}{\partial x_i^2} =$$

$$\nabla g \cdot \nabla f + g\Delta f.$$

A similar calculation gives

$$\mathrm{div}(V_2) = \nabla f \cdot \nabla g + f\Delta g.$$

Combining these equalities with Equation 61, we get

$$\int_M g\Delta f - f\Delta g = \int_M \mathrm{div}(V_1) - \mathrm{div}(V_2) =$$

$$\int_{\partial M} (V_1 \cdot \nu - V_2 \cdot \nu) = \int_{\partial M} (g\nabla f \cdot \nu - g\nabla f \cdot \nu).$$

This completes the derivation.

There are two special cases of Green's Identity worth mentioning. When f is harmonic, we get

$$\int_{\partial M} (f\nabla g \cdot \nu - g\nabla f \cdot \nu) = \int_M f\Delta g. \tag{64}$$

When f and g are both harmonic, we get

$$\int_{\partial M} g\nabla f \cdot \nu = \int_{\partial M} f\nabla g \cdot \nu. \tag{65}$$

Taking $g = 1$ gives

$$\int_{\partial M} \nabla f \cdot \nu = 0. \tag{66}$$

14.5 Averaging over Spheres and Balls

Exercise 5: Suppose M is the unit ball in $\boldsymbol{R}^n$. Apply Green's Identity to the case when $g(p) = \|p\|^2 - 1$ to deduce that

$$\int_{\partial M} 2f = \int_{M} 2nf. \tag{67}$$

In particular, when $f = 1$ show that this implies

$$\mathrm{vol}(\partial M) = n\ \mathrm{vol}(M).$$

Hint compute ∇g and Δg. Finally, combine these results to show that

$$\frac{1}{\mathrm{vol}(\partial M)} \int_{\partial M} f = \frac{1}{\mathrm{vol}(M)} \int_{M} f. \tag{68}$$

The average of a harmonic function over the unit ball is the same as the average of the function over the unit sphere!

By scaling and translation, the same result holds for any ball in $\boldsymbol{R}^n$.

Exercise 6: Define $g(p) = \|p\|^{2-n}$ on $\boldsymbol{R}^n - \{0\}$ for $n \geq 3$ and $g(p) = \log\|p\|$ on $\boldsymbol{R}^2 - \{0\}$. Prove that g is harmonic on $\boldsymbol{R}^n - \{0\}$. First hint: Use symmetry as much as possible. Second hint: If you don't want to make a brute-force calculation (even with symmetry) then flesh out the details of the following argument: Let $S(r)$ denote the sphere of radius r centered at the origin and let ν be the outward unit normal. First verify using symmetry and scaling that the integral

$$\int_{S(r)} \nabla g \cdot \nu$$

is independent of the radius $r > 0$. Conclude that the integral of Δf on any region bounded by concentric spheres centered at the origin is 0. Use this fact, and symmetry, to deduce that $\Delta f = 0$ everywhere.

Now let M be the region in $\boldsymbol{R}^n$ bounded by two concentric spheres, S_1 and S_2. Let g be the function from Exercise 6 and let f be some other harmonic function. Both f and g are harmonic on $\boldsymbol{R}^n - \{0\}$, so we have

$$\int_{\partial M} f \nabla g \cdot \nu = \int_{\partial M} g \nabla f \cdot \nu. \tag{69}$$

Consider the right hand side of the integral. Let S be one of spheres bounding ∂M. Suppose S has radius r. On S, the function g is constant. Hence

$$\int_S g \nabla f = C \int_S \nabla f = 0$$

by Equation 66. So, the right hand side of Equation 69 vanishes. This means that

$$\int_{S_1} f \nabla g \cdot \nu = \int_{S_2} f \nabla g \cdot \nu,$$

when both components S_1 and S_2 are oriented the same way. Noting that

$$\nabla g \cdot \nu = -r^{1-n}$$

on the sphere of radius r, we get

$$\frac{1}{r_1^{n-1}} \int_{S_1} f = \frac{1}{r_2^{n-1}} \int_{S_2} f. \tag{70}$$

Since S_1 and S_2 are arbitrary spheres centered at the origin, this last equation says that the average value of f is the same on all the spheres centered at the origin.

As $r \to 0$, we see that the average value of f on S_r tends to $f(0)$. This gives us the following result: $f(0)$ equals the average value of f on any sphere centered at the origin, which in turn equals the average value of f on any ball centered at the origin. More generally, the value of f at the center of a ball is equal to the average of f on that ball.

14.6 Corollaries

Theorem 14.2 (Liouville) *A bounded harmonic function on $\boldsymbol{R}^n$ is constant.*

Proof: I'll give the proof when $n = 2$. The general case is proved in the same way, using balls instead of disks. Let f be a bounded harmonic function on $\boldsymbol{R}^2$. We can scale so that $|f| \leq 1$.

Let $B(r,p)$ denote the ball of radius r about p. Let $X(r,a,b)$ denote the set of points in $B(r,a) \cup B(r,b)$ which are not in $B(r,a) \cap B(r,b)$. In other words $X(r,a,b)$ is the symmetric difference of the two balls. The area of $X(r,a,b)$ grows linearly in r. (Draw a picture!)

$$f(a) - f(b) = \frac{1}{\pi r^2} \int_{B(r,a)} f - \frac{1}{\pi r^2} \int_{R(r,b)} f =$$

$$\frac{1}{\pi r^2} \int_{B(r,a)-B(r,b)} f - \frac{1}{\pi r^2} \int_{B(r,b)-B(r,a)} f.$$

But this means that

$$|f(a) - f(b)| \leq \frac{1}{\pi r^2} \int_{X(r,a,b)} |f| \leq \frac{\pi \text{ area } X(r,a,b)}{r^2}.$$

Letting $r \to \infty$ gives $|f(a) - f(b)| = 0$. Since a and b are arbitrary, f is constant. ♠

Theorem 14.3 (Maximum Principle) *Suppose f is a non-constant harmonic function defined on an open subset U of $\boldsymbol{R}^n$. Then f cannot have a maximum at a point p in the interior of U.*

Proof: If this is false then we can find points $p, q \in U$ and a ball B such that

- f achieves a max a p.
- p is the center of B and $B \subset U$.
- $q \in B$ and $f(q) < f(p)$.

In this case $f(q') < f(p)$ for all q' sufficiently close to q. But then the average of f on B is less than $f(p)$. This is a contradiction. ♠