The Holonomy Proof of Morley's Theorem

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The purpose of these notes is to explain the holonomy proof of Morley's Theorem. I learned this proof from Peter Doyle, who says that the idea behind the proof goes back to John Smillie's construction of affine tori. The holonomy proof serves as the basis for Conway's celebrated proof of Morley's Theorem, which has several published accounts. These accounts don't focus on the holonomy idea, however.

1 Statement of the Result

Morley's Theorem refers to Figure 1. In Figure 1, the vertices of the yellow triangle are the intersection points of the angle trisectors of the big triangle. Morley's Theorem states that the yellow triangle is an equilateral triangle regardless of the shape of the outer triangle.

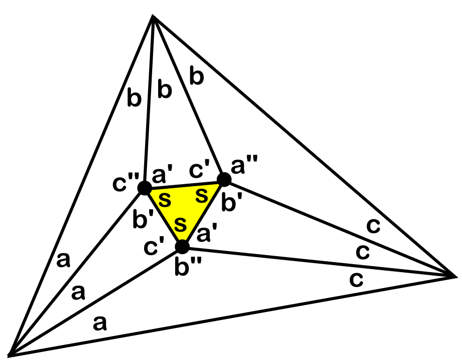

Figure 1: Morley's Theorem

The labeling of the angles is such that s is 60 degrees and $x' = x + s$ for any angle x. In particular $x'' = x + 2s$. One cannot directly calculate the angles of the configuration, but the listed values will pop out as a byproduct of the proof. One could view Figure 1 as a kind of proposal for a configuration of 7 triangles, an inspired guess.

2 The Proof Modulo a Detail

The proof boils down to showing that there are indeed 7 triangles with the given angles which fit together perfectly. Since the angles of a triangle only determine a triangle up to scale, we must show that it is possible to correctly choose the *sizes* of the triangles.

One necessary condition is that the sum of the angles around each interior vertex is $6s$. You can check this easily. However, as Figure 2 shows, just checking the angle sums around the vertices is not sufficient. The left side of Figure 2 shows a proposal for a configuration of 4 triangles. We run into trouble if we try to fit together 4 triangles around the vertex, as on the right side of the figure, so that the edges match.

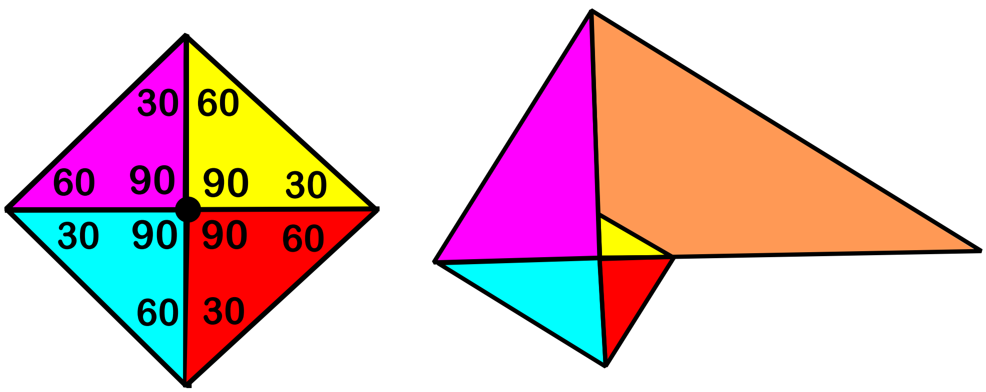

Figure 2: An impossible situation

Once we choose a size for the yellow triangle, the size of the red triangle is forced by the condition that the relevant edges have the same length. Once we have chosen the red triangle, the cyan triangle is forced, and so on. Going all the way around, we have forced a new and incompatible size for the yellow triangle. This is the large orange triangle. The *holonomy* is the ratio of sizes orange:yellow. We want to see that this bad situation does not happen for the Morley configuration. That is, we want to see that the holonomy is 1.

Figure 3, which is a repeat of Figure 1, focuses the attention on the 4 triangles going around one of the interior vertices.

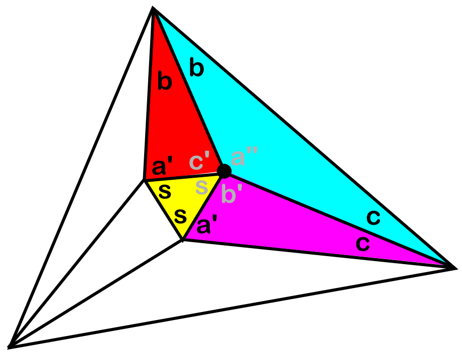

Figure 3: The Morley Configuration

Consider the angles encountered by the path which goes clockwise around the outside of this cluster of 4 triangles. The angles are:

$$s\ s\ a'\ b\ b\ c\ c\ a'$$

Note that each angle which appears in an odd position on the list also appears on an even position on the list. Call this the *parity condition*. The parity condition guarantees that we can choose 4 triangles, with the given angles, which fit together. We will give 3 proofs below.

Once we pick the 4 triangles around one of the interior vertices, we can pick the remaining 3 by going around the other 2 interior vertices. The parity condition holds at each of the other vertices as well, so everything fits together. This proves Morley's Theorem modulo the parity result.

3 Computational Proof

The **law of sines** says that a triangle with angles α, β, γ has side length $L\sin(\alpha), L\sin(\beta), L\sin(\gamma)$ for some L, with side $L\sin(\theta)$ opposing angle θ.

Let's apply this to figure 3. We choose the yellow triangle so that the yellow/red edge has length 1. By the law of sines, the successive edges of interest to us (red/cyan, cyan/magenta...) have lengths

$$\frac{\sin(a')}{\sin(b)}, \quad \frac{\sin(a')\sin(b)}{\sin(b)\sin(c)}, \quad \frac{\sin(a')\sin(b)\sin(c)}{\sin(b)\sin(c)\sin(a')}, \quad \frac{\sin(a')\sin(b)\sin(c)\sin(s)}{\sin(b)\sin(c)\sin(a')\sin(s)} = 1.$$

The last expression is the length of the *new* yellow/red edge – i.e., the holonomy. The parity condition guarantees that each term which appears on the top also appears on the bottom. Since the holonomy is 1, the 4 triangles exist.

4 Cut and Paste Proof

The Law of Sines is pretty easy to prove, but here's a proof that avoids it. Say that a *marked similarity triangle* (MST) is an equivalence class of triangles, up to similarity, with a distinguished vertex. Each MST has an invariant, which we'll call the *local holonomy*. This is the ratio A/B, where A is the length of the leading edge and B is the length of the lagging edge. Going clockwise around the marked vertex, we encounter the lagging edge, then the leading edge.

We have a finite list $T_1, ..., T_n$ of MSTs such that the angle sum around the marked points is 360 degrees, and we want to know if we can fit together an actual list of triangles $t_1, ..., t_n$, with t_k being a representative of T_k for all k. If this happens, call $\{T_k\}$ an *isometric list*. The holonomy is the product of the local holonomies, so the list is isometric if and only if the product of the holonomies is 1. In particular, $\{T_k\}$ is isometric if and only if every permutation of $\{T_k\}$ is isometric.

Consider the special case where our list is such that an MST appears on the list if and only if its reflected copy appears on the list. Well call this the *reflection property*. When $\{T_k\}$ has the reflection property, we can permute the list so that each MST appears next to its reflected copy. This permuted list is isometric by symmetry. Hence, the original list is isometric as well.

Now suppose that $\{T_k\}$ satisfies the parity condition. To make the argument go more cleanly, let's suppose that each MST is nice in the sense that the perpendicular from the marked point to the opposite edge stays inside the triangle. For Morley's Theorem, the relevant lists are all clean if $a', b', c' \leq 90$, and this happens when the outer triangle is right or acute. When all the MSTs are nice, we can drop all the perpendiculars from the marked points. By the parity condition, this produces a list of $2n$ MSTs with the reflection property. This new list is isometric, and hence so is the original list.

With some effort, one can modify this proof so that it also works when some of the MSTs are not nice. The next section gives a similar kind of cut-and-paste proof which works in all cases.

5 Second Cut and Paste Proof

We'll consider a setup similar to the one in the previous proof, but this time we will drop the condition that the sum of the angles around the marked points is 360 degrees. Suppose that we have a periodic and bi-infinite list $\{T_n\}$ of MSTs. Corresponding to this list, there is a bi-infinite periodic list $\{t_n\}$ of triangles which fit together around the origin. Here t_n is a representative of T_n. There is a similarity h fixing the origin such that $h(t_k) = t_{t+N}$ for all k. Here N is the length of the period. The map h is called the holonomy. It is independent of the choice of $\{t_n\}$, a choice which is unique up to scale. We call $\{T_n\}$ *isometric* if h is an isometry. The scale factor for h is the product of the local holonomies over one period.

Permuting the MSTs and using the product formula for the scale, we reduce to the case when the angle list is just:

$$...(a_1a_2)(a_2a_3)...(a_{N-1}a_N)(a_Na_1)...$$

The rest of the proof goes by induction on N. When $N = 2$ the result follows from symmetry. At least one of the angles in the angle list is acute, and so we may insert an isosceles triangle, if necessary, to guarantee that $N \geq 4$ is even. Figure 4 shows a cut-and-paste operation which replaces our list of length N by a new list of length $N - 2$. The new list is isometric if and only if the old list is isometric.

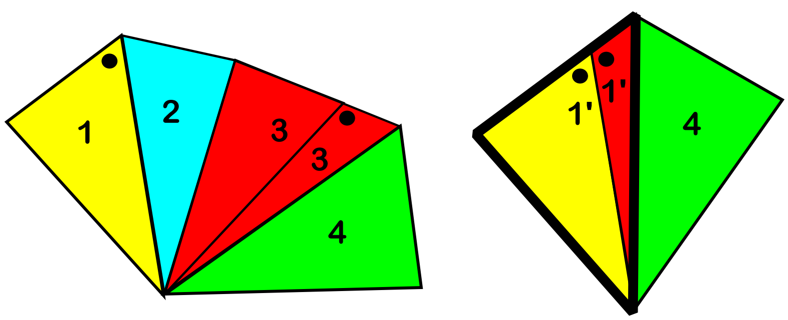

Figure 4: Cutting out 2 triangles

Arrange so that T_2 has the smallest angle around the marked vertex. The MSTs $T_1, T_2, T_3, T_4, ...$ are replaced by $T_1', T_4, ...$ Here T_1' is the union of T_1 and a rotated copy of $T_3 - R(T_2)$, where $R(T_2)$ is the reflection of T_2 across $T_2 \cap T_3$. The angles with the dots are complementary, which makes everything fit together.