Two Geometric Pictures of Farey Addition

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The *Farey sum* of two rational numbers a/b and c/d is

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}. \tag{1}$$

At first glance this operation looks unpromising: It seem to be a mistaken way to add fractions. However, Farey addition is a well-studied concept in number theory. See e.g. [**HW**, §III]. Farey addition has connections to such topics as continued fractions and the modular group. It is also the basic structure underpinning the recently created and celebrated quantum rationals [**MO**].

In this note I will give two interpretations of Farey addition, one in terms of an operation on certain pairs of disks and one in terms of Pappus's theorem, a famous theorem from projective geometry. The disk interpretation is extremely well known. See for instance [**W**, pp. 88-89]. For all I know, the Pappus interpretation has also previously been discovered, but I am pretty certain that the deeper connections between Pappus's Theorem and the modular group originated in my paper [**S1**]. My recent paper [**S2**] gives a nice exposition of [**S1**] and also goes more deeply into the topic. At the end of this note, I will briefly indicate how the elementary material here leads naturally into deeper waters.

Farey Partners: To make the connections work perfectly I will always work with reduced fractions a/b. This is to say that a, b are coprime. Also, I will only apply Farey addition to pairs a/b and c/d such that $|ad - bc| = 1$. I will call such fractions *Farey partners*. This is the usual constraint one sees

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when working with continued fractions and the modular groups.

The Disk Interpretation: Let's interpret the fraction a/b as the disk $D(a/b)$ having center

$$\left(\frac{a}{b}, \frac{1}{2b^2}\right) \tag{2}$$

and radius $1/(2b^2)$. These disks lie in the upper half plane and are tangent to the x-axis.

Lemma 0.1 *$D(a/b)$ and $D(c/d)$ are tangent when a/b and c/d are Farey partners.*

Proof: The square distance between the centers of the disks is

$$d^2 = \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 + \left(\frac{a}{b} - \frac{c}{d}\right)^2 = \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 + \frac{1}{b^2d^2}.$$

At the same time, the square of the sum of the radii is

$$s^2 = \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2.$$

When you expand out the expressions you see that $d^2 = s^2$. Since $d, s > 0$ we have $d = s$. ♠

Note that $e/f := a/b \oplus c/d$ is a Farey partner with both a/b and c/d. This means that the disk $D(e/f)$ is mutually tangent to the $D(a/b)$ and $D(c/d)$ and the x-axis. Thus, If we start with two disks which are Farey partners, then Farey addition amounts to inserting the unique disk that lies in the interstitial region above the x-axis and between the two disks. Figure 1 shows this in action.

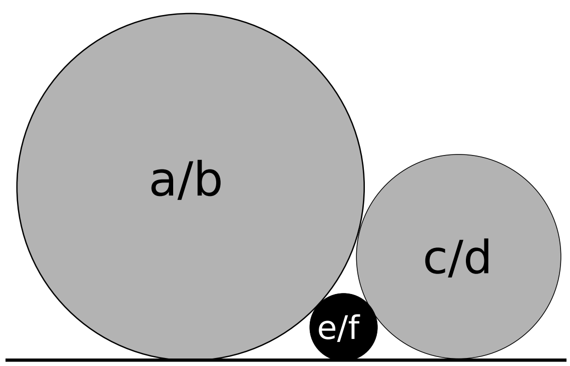

Figure 1: The disk interpretation of Farey addition

The Pappus Interpretation: Many geometry textbooks discuss Pappus's famous theorem. See e.g. [**H**]. On [**HC**, p 103], D. Hilbert and S. Cohn-Vossen refer to the configuration underlying Pappus's Theorem, namely the one in Figure 2, as "the most important configuration in all of geometry".

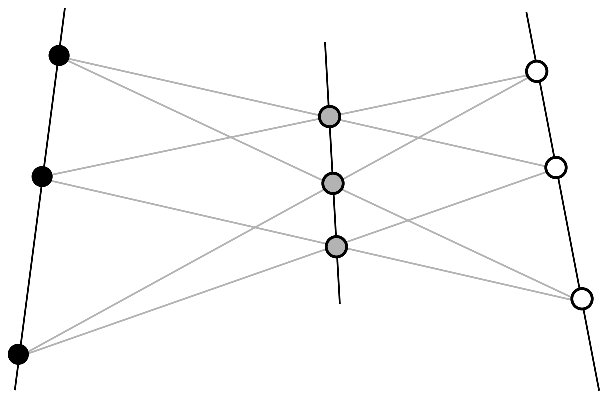

Figure 2: Pappus's theorem.

Theorem 0.2 (Pappus) *If the black points are collinear and the white points are collinear then so are the grey points.*

Proof: Here is a sketch of a direct computational proof. We can encode the point (x, y) as the scale equivalence class of the vector $(x, y, 1)$ in $\boldsymbol{R}^3$. Likewise we can encode the line $ax + by + 1 = 0$ as the scale equivalence class of the vector $(a, b, 1)$. These coordinates are known as *homogeneous coordinates*. In homogeneous coordinates, the intersection of two lines is given by the cross product. Likewise, the line through a pair of points is given by the cross product. Three vectors represent collinear points if and only if, when you put them as columns in a 3×3 matrix, you get a matrix with zero determinant. Equipped with these computational tools, you just coordinatize the white and black points, take a bunch cross products, and then see that one final determinant vanishes. ♠

Now we take a very special case of Pappus's Theorem. (In this special case, the conclusion of Pappus's theorem just follows from symmetry.) We represent a/b by the vertical line segment $L(p/q)$ whose length is $1/b$ and whose midpoint is $(a/b, 0)$. We think of the endpoints and the midpoint as the three special points on $L(a/b)$. This, $L(a/b)$ is really a *marked line segment*, meaning a line segment with 3 distinguished points. Here is the statement parallel to Lemma 0.1

Lemma 0.3 *Suppose a/b and c/d are Farey partners. When we perform Pappus's Theorem starting with $L(a/b)$ and $L(c/d)$ the new marked line segment created is $L(e/f)$.*

Proof: This is a direct calculation, best done in homogeneous coordinates. The marked points of $L(a/b)$ are represented by vectors V_{-1}, V_0, V_1, where $V_j = (a/b, j/b, 1)$ The marked points of $L(c/d)$ are represented by vectors W_{-1}, W_0, W_1, where $W_j = (c/d, j/d, 1)$

Let L' be the marked segment produced by our operation. We want to see that $L' = L(e/f)$. By symmetry, L' is a vertical line segment centered on the x-axis. The middle point of L' is represented by the vector

$$(V_{-1} \times W_1) \times (V_1 \times W_{-1}) = \frac{ad - bc}{b^2 d^2}(a + c, 0, b + d).$$

This vector represents $(e/f, 0)$. The top point of L' is represented by the vector

$$(V_0 \times W_1) \times (V_1 \times W_0) = \frac{ad - bc}{b^2 d^2}((a + c, 1, b + d).$$

This vector represents $(e/f, 1/f)$. The calculation for the bottom point of L' follows from the top-point calculation and symmetry. ♠

Figure 3 shows Lemma 0.3 in action for $a/b = 1/3$ and $c/d = 1/2$.

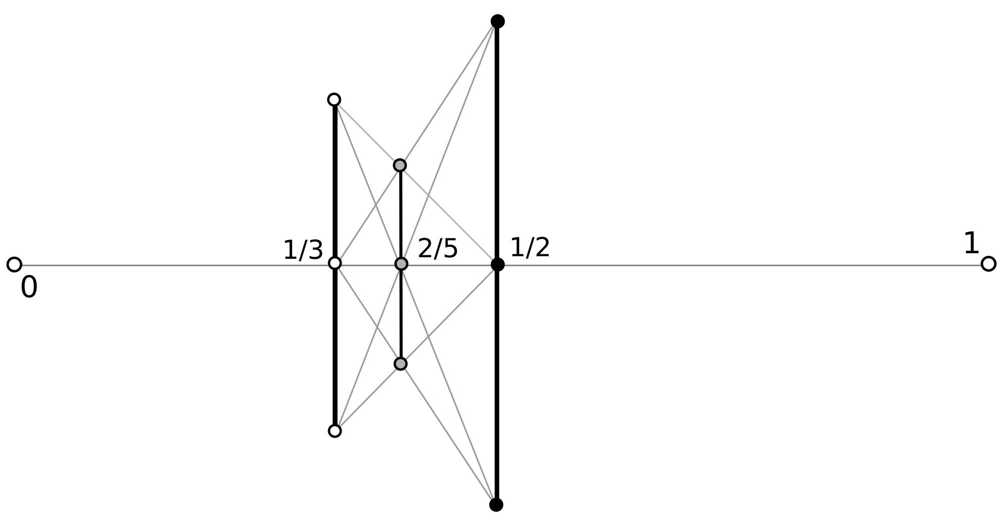

Figure 3: Pappus Interpretation of Farey Addition

For the sake of drawing nice pictures, we point out an extra symmetry in the Pappus interpretation. We might have chosen an arbitrary $\lambda > 0$ and then defined $L(a/b)$ to be the line segment whose length is λ/b and whose center is $(a/b, 0)$. Doing this has the effect of vertically stretching or compressing the picture, and it has no effect on the essential properties of the construction. When it comes time to show our big picture below, we will compress the picture in this way, so that everything fits nicely on the page.

Constructing the Rationals: Both the disk interpretation and the Pappus interpretation lend themselves to a recursive construction of all the rationals between 0 and 1. To use disks, start with $D(0/1)$ and $D(1/1)$ and then recursively fill in the interstitial regions. To use Pappus's Theorem, start with $L(0/1)$ and $L(1/1)$ and then recursively apply Pappus's Theorem. Figure 4 shows the two constructions, one on top of the other. They line up perfectly.

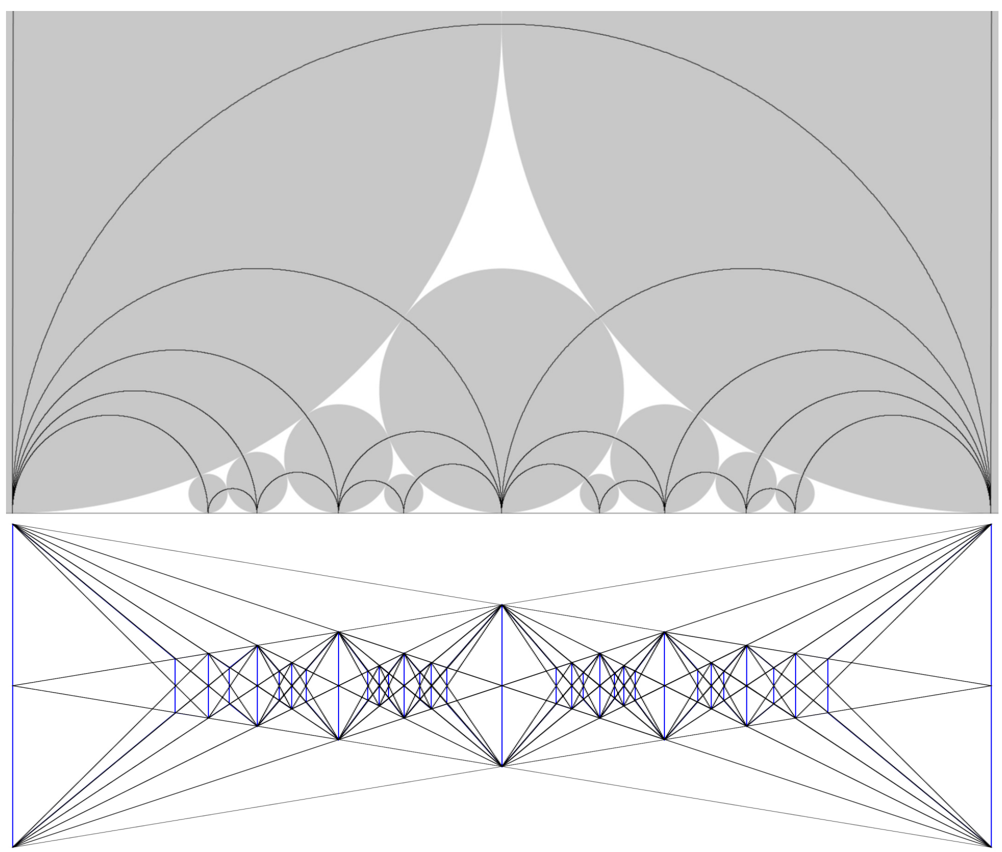

Figure 4. Pappus Interpretation of Farey Addition

The top part of Figure 4 has some extra stuff in it. I have joined Farey partners by semi-circular arcs. These arcs go through the tangency points of the grey disks.

Deeper Waters: The circular arcs in Figure 4 are geodesics in the upper half plane model of the hyperbolic plane. See e.g. [**K**] and [**T**] for extensive discussions of the hyperbolic plane and see e.g. [**Se2**] for a drawing of these geodesics in the disk model of the hyperbolic plane. (In the disk model you can grasp the whole pattern more easily.) These geodesics are the edges of the *Farey tesselation*, a totally symmetric tiling of the hyperbolic plane by ideal triangles.

At the same time, the pattern of disks extends outside the portion shown in Figure 4 to fill up the entire hyperbolic plane. This extended pattern of disks is often referred to as the *horodisk packing dual to the Farey tesselation*. Concretely, you get the extension from (the fully filled-in version of) the top half of Figure 4 by first translating the whole picture to the left and right by all integer translations and then adding in the "infinite horodisk" consisting of the set of all points above the line $y = 1$. In this extension, the disks and the geodesics are everywhere related just as in Figure 4: Each geodesic goes through a tangency point between a pair of disks.

In the upper half-plane model, the Farey tesselation and the dual horodisk packing are simultaneously preserved by the action of the modular group, $PSL_2(\boldsymbol{Z})$. (This is the group of 2×2 integer matrices of determinant 1 modulo the relation that $M \sim -M$.) So, the circle interpretation above fits in naturally with hyperbolic geometry and modular group symmetry. See e.g. [**Se1**] and [**Se2**] for a discussion of how the modular group, continued fractions, and hyperbolic geometry are related.

What about the Pappus interpretation? First of all, one can also extend the construction "outwards". This will produce a larger pattern of points and lines. This pattern, properly understood, lives in the flag variety over the projective plane. The flag variety is the space of pairs (p, ℓ) where p is a point in the projective plane and ℓ is a line in the projective plaene containing p. The natural group of symmetries here is the group of isometries of the 5 dimensional symmetric space $X = SL_3(\boldsymbol{R})/SO(3)$. So, once we make the extension, we get an isomorphic copy of the modular group sitting inside the isometry group of X.

But this is just the beginning! The thing about Pappus's Theorem is that it applies to many different starting configurations and not just the one I chose. You can change the initial conditions, using some other pair of "marked line segments" in place of $L(0/1)$ and $L(1/1)$. When you do this, you see interesting things emerge. Figure 5 shows the same kind of iterative construction starting from a different "seed" of 2×3 points. I have taken some liberties with the coloring to get a striking image. What happens when you change the initial conditions is that some of the segments tilt and their distinguished midpoints move up and down.

As with the "symmetric case" shown in Figure 4, the picture here extends outwards. The set of midpoints of the segments turn out to be dense in a continuous fractal loop. There is still a group of symmetries of the whole pattern. It is algebraically isomorphic to the modular group but geometrically quite different. It is known as a *deformation* of the modular group inside the isometry group of X. Even the geodesics in Figure 4 have an interpretation here: They form a certain pattern of geodesics in X with the same kind of asymptotic properties as the Farey pattern in the hyperbolic plane, except that somehow they are bent inside X. The picture is very simular to what is called a *pleated surface*.

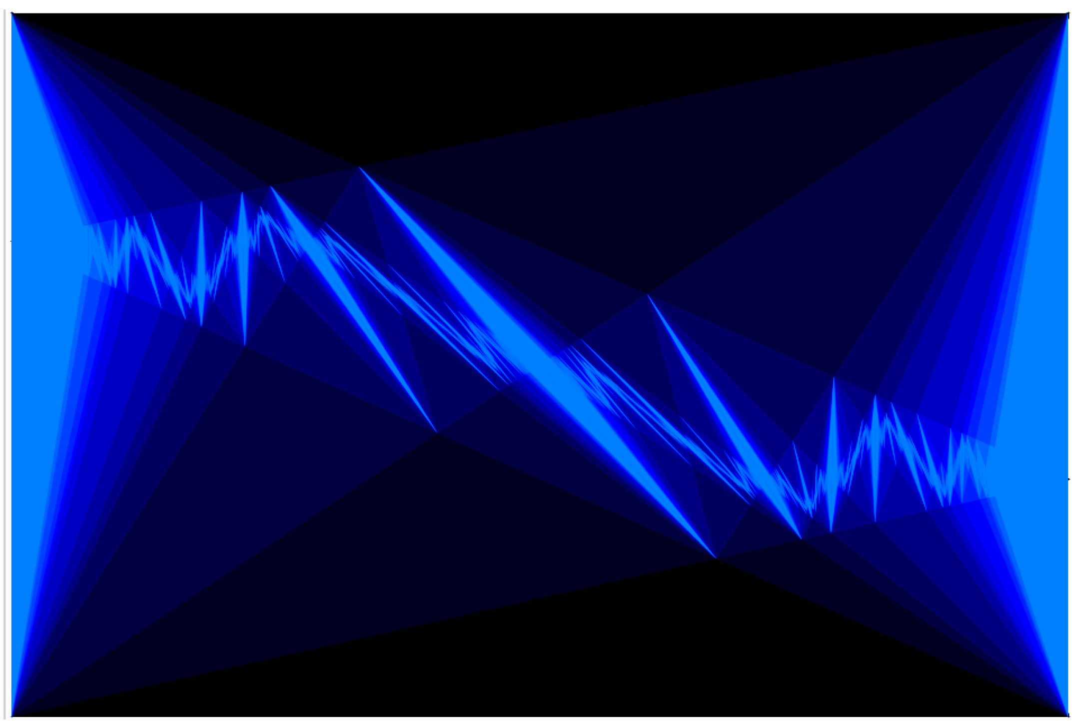

Figure 5. An iteration of Pappus's Theorem with a different seed.

What I am saying is that you can start with the hyperbolic geometry picture suggested by the top of Figure 4, then switch interpretations to the Pappus picture. Then you can perturb the initial conditions and extend outward. When you do this, and suitably interpret the construction, the whole thing pops out into a kind of pleated and yet totally symmetric pattern of geodesics sitting inside the 5 dimensional space X. It is hard to picture these patterns directly, but I imagine a being gifted with the right kind higher dimensional sight would find them quite beautiful. If you want to know the details, see [**S2**].

Let me close this note with a question. The iterated Pappus construction and the quantum rational construction both give enhanced extensions of the same kinds of mathematics – rational numbers, continued fractions, the modular group, etc. Is there some way to marry these two ideas?

1 References

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