The Optimal Twisted Paper Cylinder

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Abstract

A smooth twisted paper cylinder of aspect ratio λ is an isometric embedding of a $1 \times \lambda$ cylinder into $\boldsymbol{R}^3$ such that the images of the boundary components are linked. We prove that for such an object to exist we must have $\lambda > 2$ and that this bound is sharp. We also show that any sequence of examples having aspect ratio converging to 2 must converge, up to isometries, to a certain 4-fold wrapping of a right-angled isosceles triangle.

1 Introduction

This paper is a sequel to my paper [**S**] in which I prove the optimal paper Moebius band conjecture of Halpern and Weaver [**HW**] from 1977. The main result of [**S**] is that a smooth embedded paper Moebius band must have aspect ratio greater than $\sqrt{3}$. Actually, the minimum aspect ratio question is already raised in a 1930 paper of M. Sadowsky [**Sa**] and again in a 1962 paper of W. Wunderlich [**W**]. (See [**T**] and [**HF**] for English translations of [**W**] and [**Sa**] respectively.)

This solution of the optimal Moebius band conjecture raises new questions. Suppose we only consider paper Moebius bands made from $1 \times \lambda$ strips of paper which are given at least $2k+1$ twists. Then there is some smallest λ_k such that $\lambda \geq \lambda_k$ for all examples. The main result of [**S**] combines with the work in [**HW**] to show that $\lambda_0 = \sqrt{3}$. Based on a beautiful polygonal 3-twist Moebius band of aspect ratio 3 that Brienne Brown made, I think

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that $\lambda_1 = 3$. (I used to think that the hexaflexagon, which has aspect ratio $3\sqrt{3}$, was best. Brienne's example surprised me.) For larger k I do not have a guess. These questions involve twisting a strip of paper an odd number of times. In this paper we will consider the even case.

We call two subsets $S_1, S_2 \subset \boldsymbol{R}^3$ *entangled* if for every linear projection Π from $\boldsymbol{R}^3$ onto a plane the sets $\Pi(S_1)$ and $\Pi(S_2)$ intersect. This happens, for instance, if S_1 and S_2 are two components of a non-trivial link. Define

$$\Gamma_\lambda = ([0,1] \times [0,\lambda])/\sim, \qquad (t,0) \sim (t,\lambda). \tag{1}$$

The boundary $\partial\Gamma_\lambda$ consists of 2 geodesic circles, $\partial_1\Gamma_\lambda$ and $\partial_2\Gamma_\lambda$.

Say that an *entangled paper cylinder of aspect ratio* λ is a smooth isometric mapping $I : \Gamma_\lambda \to \boldsymbol{R}^3$ such that $I(\partial_1\Gamma_\lambda)$ and $I(\partial_2\Gamma_\lambda)$ are entangled. Here, an *isometric mapping* is one in which the differential dI is an isometry at each point. We do not require that I is an embedding. In case I is an embedding, and the two components of $I(\partial\Gamma_\lambda)$ are linked, we call our object a *twisted paper cylinder*. A twisted paper cylinder is a special case of an entangled paper cylinder.

The question is: How small can one make λ? This is an analogue of the question discussed above for Moebius bands. As far as I know, this question for cylinders has not appeared in the literature, though I admit that I do not know the literature well. The theorems I prove here completely answer this question.

Theorem 1.1 *An entangled paper cylinder has aspect ratio greater than* 2.

Theorem 1.2 *For any* $\epsilon > 0$ *there exists a twisted paper cylinder of aspect ratio less than* $2 + \epsilon$.

Figure 1 (left) shows a pattern of fold segments on a 1×2 strip of paper and Figure 2 (right) shows the picture of what you get when you fold the paper along these lines. If you fold the pieces in the correct direction and then tape along the left edge, you will get a paper cylinder of aspect ratio 2. The lettering on the left tells you the ordering of the faces when the strip is folded. Thus, if you fold the thing up, lay it on the table, and poke it with a needle, the needle will encounter faces A, B, C, D in that order. The coloring indicates the sense: The red faces are up when the thing is sitting on the table and the blue faces are down. The other side of the strip would be colored in the opposite way and would have the same lettering.

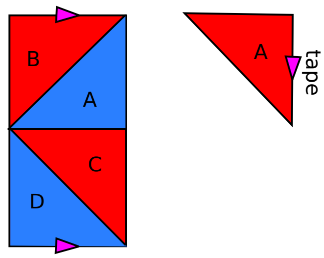

Figure 1: The right-isosceles cylinder.

It is not quite embedded, but we will show how you can find arbitrarily small perturbations which are smooth twisted paper cylinders. These perturbations are responsible for Theorem 1.2. I encourage you to make one of these. It is rather difficult to picture the folding pattern without making it. As Noah Montgomery pointed out to me after reading an earlier draft of this paper, there are actually two distinct folding patterns which give this shape; the other one is quite similar to Figure 1. We call either of these examples *right isosceles cylinders*.

We will also show that the right isosceles cylinders are the only limits of a sequence of minimizers:

Theorem 1.3 *Suppose* $I_n : \Gamma_{\lambda_n} \to \boldsymbol{R}^3$ *is a sequence of entangled paper cylinders such that* $\lambda_n \to 2$*. Then a subsequence of* $\{I_n\}$ *converges uniformly, up to isometries, to a map whose image is a right isosceles cylinder.*

Paper Moebius bands are closely related to *folded ribbon knots*, and the right isosceles cylinders may be interpreted as folded ribbon knots. Informally, a folded ribbon knot is a planar polygonal version of a paper Moebius band or cylinder. See [**DL**] and [**D**] for a wealth of information about these. With a bit of soft work concerning smooth approximations of folded ribbon knots (along the lines of the *pseudofolds* in [**HW**]) our results resolve the case $n = 1$ of [**DL**, Conjecture 39].

The proofs of Theorems 1.1 and 1.3 are similar in spirit to what we do in [**S**] but different in the details. The proof of Theorem 1.2 is similar in spirit to what is done in [**HW**] but confined to a much more specific situation. Compare also the construction in [**Sa**]. I prove all three results in the next chapter.

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2 The Proofs

2.1 The Bends

Let $I : \Gamma_\lambda \to \Omega$ be a smooth twisted paper cyclinder. A *bend* on Ω is a straight line segment having one endpoint in one boundary component of Ω and one endpoint in the other. The following lemma is, I think, a classic result. Since I only read about this result in connection with Moebius bands, I will give the argument in somewhat more detail than I did in [**S**].

Lemma 2.1 *Ω has a continuous partition into bends.*

Proof: Here is a sketch. Let $U \subset \Omega$ denote the nonempty subset where Ω has nonzero mean curvature. The same discussion on the bottom of p 46 in [**HW**] applies in the cylinder case; it is completely local. The result is that U is partitioned into bends which are parallel to the directions of zero extrinsic curvature on the surface. These directions exist by Gauss's Theorema Egregium.

The bends of U vary continuously because they are disjoint and all have length bounded from below (by 1 unit). Hence the partition of U by bends extends to give a continuous partition of the closure $\overline{U}$ by bends. The complementary regions of $\Omega - \overline{U}$ are each flat open trapezoids. Two opposite parallel sides of each trapezoid τ lies in $\partial\Omega$ and the other two sides are bends of $\overline{U} - U$. These bends can be continuously extended to partition τ. Doing this for all such τ we extend the bend partition of $\overline{U}$ to a bend partition of Ω. Again, the bends vary continuously because they are disjoint and all have length at least 1. ♠

2.2 Balanced Pairs of Bends

The bends on Ω are naturally parametrized by the $R/\lambda\boldsymbol{Z}$. Formally, one takes the pre-image of a bend and looks where this line segment on Γ_λ intersects the midline, a loop we identify with $\boldsymbol{R}/\lambda\boldsymbol{Z}$. Suppose that (u, v) is an ordered pair of bends on Ω. These bends partition each component of $\partial\Omega$ into two arcs. We call (u, v) a *balanced pair* if each arc of each partition has length $\lambda/2$. This is just to say that each arc partition consists of arcs having the same length.

Lemma 2.2 *Ω has a balanced pair of arcs.*

Proof: The proof here is very much like the proof of Lemma T in [**S**]. It is really just the Borsuk-Ulam Theorem. Let Υ denote the cylinder obtained by deleting the diagonal from $(\boldsymbol{R}/\lambda\boldsymbol{Z})^2$. Each pair $(x,y) \in \Upsilon$ gives rise to an ordered pair of bends, and *vice-versa*.

We compactify Υ by adding two boundary components. A point (x,y) is near one boundary component if y is just ahead of x in the cyclic order on $\boldsymbol{R}/\lambda\boldsymbol{Z}$. A point (x,y) is near the other boundary component if x is just ahead of y in the cyclic order on $\boldsymbol{R}/\lambda\boldsymbol{Z}$. Let $\overline{\Upsilon}$ be this compactification. Now we have two cylinders, Ω and $\overline{\Upsilon}$, and the argument involves both.

Let F and G denote the two components of $\partial\Omega$. For $j=1,2$ let $f_j(x,y)$ denote the lengths of the two components of $F-u-v$. The ordering of the two components of $F-u-v$ can be chosen consistently so that these functions vary continuously and extend continuously to $\partial\overline{\Upsilon}$. We let

$$f = f_1 - f_2.$$

The function f equals $-\lambda$ on one component of $\partial\overline{\Upsilon}$ and $+\lambda$ on the other. We define the function g with respect to G in the same way. We note one fundamental symmetry of these functions. Let $\Sigma(x,y)=(y,x)$. Then we have $f\circ\Sigma = -f$ and $g\circ\Sigma=-g$.

To finish the proof we just have exhibit a point $(x,y)\in\Upsilon$ having the property that $f(x,y)=g(x,y)=0$. We will suppose this does not happen and derive a contradiction. If f,g do not simultaneously vanish on Υ then they also do not simultaneously vanish on $\overline{\Upsilon}$. This means that the functions $A=(f,g)$ and $B=(\sqrt{2})A/\|A\|$ are continuous functions on $\overline{\Upsilon}$.

Define a *crosscut* to be any curve which joins one boundary component of $\overline{\Upsilon}$ to the other component. If γ is such a crosscut, then $B(\gamma)$ is a curve on the unit circle which connects $(1,1)$ to $(-1,-1)$. We always orient $B(\gamma)$ to that it starts at $(1,1)$ and ends at $(-1,-1)$. We let $w(\gamma)$ denote the number of times γ winds counter-clockwise around the circle. Then $w(\gamma)$ is a half-integer.

Every two crosscuts are homotopic relative to $\partial\overline{\Upsilon}$, so $w(\gamma)$ is independent of γ. On the other hand, note that $B(\Sigma(\gamma))=-B(\gamma)$. So, when we again orient $B(\Sigma(\gamma))$ from $(1,1)$ to $(-1,-1)$ we find that $w(\Sigma(\gamma))=-w(\gamma)$. This is a contradiction because $\Sigma(\gamma)$ is just another crosscut. ♠

2.3 The Line Segment Lemma

We are going to take a little detour and prove a result about line segments in the plane. Let $\ell(\cdot)$ stand for arc length. An arc is *rectifiable* if it has finite length. We are only interested in the smooth case, but there is no harm in considering the extra generality.

Lemma 2.3 *Let A and B be two segments in the plane, each having length at least 1. Let A_1, A_2 be the endpoints of A and let B_1, B_2 be the endpoints of B. For $j = 1, 2$ let C_j be a rectifiable arc that connects A_j to B_j. If $C_1 \cap C_2 \neq \emptyset$ then $\ell(C_1) + \ell(C_2) \geq 2$.*

Proof: The left side of Figure 2.1 shows the case where A and B intersect. The right side shows the case where they do not. The argument is the same in both cases. (We leave it to the reader to imagine the case $A = B$.) Let x be a point where C_1 and C_2 intersect. We do not increase $\ell(C_j)$ if we replace C_j by a bigon which connects x to A_j and B_j. Let $C_{j,A}$ and $C_{j,B}$ denote the two segments of this bigon. Figure 2.1 shows the situation.

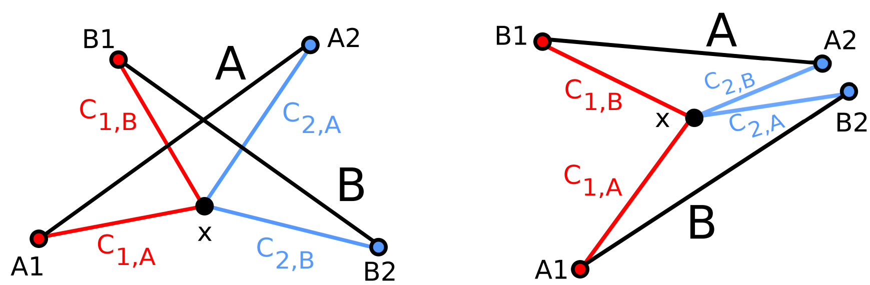

Figure 2.1: The diagram for the crossing case

We have

$$\ell(C_1)+\ell(C_2) = \ell(C_{1,A})+\ell(C_{2,A})+\ell(C_{1,B})+\ell(C_{2,B}) \geq \ell(A)+\ell(B) \geq 2. \quad (2)$$

This completes the proof. ♠

Remark: In case we have equality in Equation 2 and $A \neq B$ the segments A and B must intersect, and C_1, C_2 must be bigons which intersect at the point $x = A \cap B$. Moreover, A and B must have length 1. If we are close to equality in Equation 2 the picture must almost look like this limiting case. We leave the reader to contemplate the case $A = B$.

2.4 Proof of Theorem 1.1

Now we go back to our entangled paper cylinder Ω. Let (u, v) be our balanced pair. In the embedded case we have $u \neq v$ but in the immersed case we have to worry that $u = v$. We treat this irritating special case at the end. So, for now assume that $u \neq v$.

There is a unique pair (U, V) of parallel planes such that $u \in U$ and $v \in V$. If u and v are parallel then they are co-planar and $U = V$ is this common plane. Otherwise U and V are both perpendicular to $\overrightarrow{u} \times \overrightarrow{v}$, the cross product of vectors parallel to u and v respectively. We rotate to that U and V are parallel to the XY-plane and we project the whole picture into the XY plane. Let Π be this planar projection.

Let $A = \Pi(u)$ and $B = \Pi(v)$. Notice that

$$\ell(A) = \ell(u) \geq 1, \qquad \ell(B) = \ell(v) \geq 1.$$

Let F and G be the two boundary components of Ω. By assumptions the sets $\Pi(F)$ and $\Pi(G)$ intersect. In particular there are arcs $F' \subset F$ and $G' \subset G$, each joining an endpoint of u to an endpoint of v, such that $C_1 = \Pi(F')$ and $C_2 = \Pi(G')$ intersect. But then

$$\lambda = \ell(F') + \ell(G') \geq \ell(C_1) + \ell(C_2) \geq 2. \tag{3}$$

The equality comes from the fact that (u, v) is balanced. The first inequality comes from the fact that Π is distance non-increasing. The second inequality comes from the Line Segment Lemma.

This shows that $\lambda \geq 2$. If $\lambda = 2$ we have equality in Equation 3. This forces u, v to be coplanar because otherwise Π would not preserve length along F' and G'. Moreover A and B would have to intersect, as we remarked after the proof of the Line Segment Lemma. This means that u and v intersect as well. In case Ω is embedded this is a contradiction. Hence $\lambda > 2$ in the embedded case. In the general case we will show below that $\lambda = 2$ forces Ω to be the right isosceles cylinder, and this is not smooth. So, even in the immersed case, we have $\lambda > 2$.

It only remains to consider the possibility that $u = v$. In this case the planes U and V are not unique. But we simply take $U = V$ to be any plane that contains $u = v$. We let Π be projection onto this plane. Once we have done this, the rest of the argument works the same way as when $u \neq v$.

2.5 Proof of Theorem 1.2

The proof of this theorem is very much like what is done in [**Sa**] and (in more detail) in [**HW**]. This process is also discussed more simply and informally in [**FT**]. The basic idea is to build the limiting shape first and then relax the shape a bit and round out the folds. I think of this as a bit like making the thing and then soaking it overnight in water.

Here is a more formal description. First of all, using smooth bump functions one can easily make a U-shaped curve. This curve agrees with line parallel line segments at either end and then curves around to join these line segments. Call this curve U. The product $U \times [a, b]$ is an isometrically embedded rectangle. Next, take the limiting model (which you should try enough times to make that you are convinced that it exists) and separate out the 4 triangles a little bit. Join the newly created edges of this stack of triangles by the U-shaped rectangles.

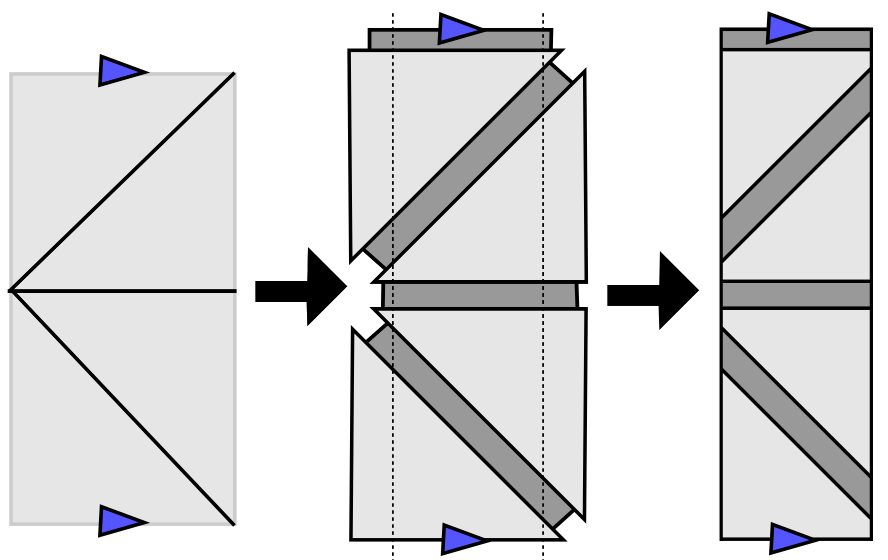

Figure 2.2: Perturbing the limiting model

The new object will be an embedded smooth twisted cylinder whose boundary is not quite totally geodesic. Figure 2.2 shows schematically what this looks like. Now just trim off the rough edges. This gives you a smooth embedded twisted cylinder with slightly larger λ. With this procedure you can make λ as close to 2 as you like. This completes the proof of Theorem 1.2.

2.6 Proof of Theorem 1.3

Now we suppose that we have an entangled paper Moebius band $I : \Gamma_\lambda \to \Omega$ with λ just a tiny bit bigger than 2. To simplify our argument we just assume that I is a continuous immersion, the limit of smooth maps, and $\lambda = 2$. In this case we have the equality we have been considering above.

The coplanar bends u and v intersect at a point x. (If $u = v$ we take x to be the midpoint of this common segment.) We claim that x is an endpoint of u and v. (In particular $u \neq v$.) If not then there is some bend of Ω which connects x to a point on $u \cup v$. But then this bend has length less than 1, a contradiction.

The only possibility is that u and v intersect at an endpoint. Moreover, the map I is a linear isometry on each of the two arcs F_1, F_2 of $F - u - v$ and on each of the two arcs G_1, G_2 of $G - u - v$. (Again, F and G are the components of $\partial\Gamma_2$.) Also, we mention again that $\ell(u) = \ell(v) = 1$. This means that the flat cylinder Γ_2 has a partition into squares, and each of these squares has a partition into two triangles, and the restriction of I to each triangle is an isometry. The reason for this last conclusion is that this restriction is determined by its action on $\partial\Gamma_2$, and this is an isometry. Our triangles must be right-isosceles because we get them by adding a diagonal to a unit square. This shows that u and v are perpendicular, and the image Ω is just a right-isosceles triangle of unit side length.

So, up to the folding pattern, we have the same picture as in Figure 1 or as in the closely related variant. A bit of experimentation shows that there is really just two ways to fold the strip in Figure 1, along its bending lines, to produce a twisted cylinder. So, our map I must be the one o the other of these maps which give the right isosceles cyinder. In order to actually get a limit, we have to pass to a subsequence so as to pick out one or the other of the right isosceles cylinders.

This completes the proof of Theorem 1.3. This also takes care of the one detail left over from our proof of Theorem 1.1, namely that $\lambda = 2$ cannot happen in the smooth immersed case.

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