Dehn's Dissection Theorem

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1 The Result

A *polyhedron* is a solid body whose boundary is a finite union of polygons, called *faces*. We require that any two faces are either disjoint, or share a common edge, or share a common vertex. Finally, we require that any edge common to two faces is not common to any other face.

A *dissection* of a polyhedron P is a description of P as a finite union of smaller polyhedra,

$$P = P_1 \cup ... \cup P_n, \tag{1}$$

such that the smaller polyhedra have pairwise disjoint interiors. Note that there is not an additional assumption, say, that the smaller polyhedra meet face to face.

Two polyhedra P and Q are *scissors congruent* if there are dissections $P = P_1 \cup ... \cup P_n$ and $Q = Q_1 \cup ... \cup Q_n$ such that each P_k is isometric to Q_k. Sometimes, one requires that all the isometries are orientation-preserving, but in fact and two shapes that are scissors congruent *via* general isometries are also scissors congruent *via* orientation preserving isometries. (This little fact isn't something that is important for our purposes.)

One could make the same definition for polygons, and any two polygons of the same area are scissors congruent to each other. (J. Dupont says that W. Wallace first formally proved this in 1807.) Hilbert asked if every two polyhedra of the same volume are scissors congruent to each other. In 1901, Max Dehn proved the now-famous result that the cube and the regular tetrahedron (of the same volume) are not scissors congruent. We're going to give a proof of this result.

2 Dihedral Angles

The *dihedral angle* is an angle we attach to an edge of a polyhedron. To define this angle, we rotate so that the edge in question is vertical, and then we look directly down on the polyhedron. The two faces containing our edge appear as line segments, and the dihedral angle is the angle between these line segments. We will measure dihedral angles in such a way that a right angle has measure 1/4. All the dihedral angles of a cube are 1/4.

All edges of a regular tetrahedron have the same dihedral angle. We're going to prove that this common angle is irrational. Geometrically, this is the same as saying that one cannot fit finitely many tetrahedra precisely around an edge, even if these tetrahedra are permitted to wrap around more than once before closing back up.

We will place our tetrahedron in space so that one edge is vertical. Rather than work in $\boldsymbol{R}^3$, it is useful to work in $\boldsymbol{C} \times \boldsymbol{R}$, where $\boldsymbol{C}$ is the complex plane. This is a nice way to distinguish the vertical direction. Consider the complex number

$$\omega = \frac{1}{3} + \frac{2\sqrt{2}}{3} i. \tag{2}$$

Note that $|\omega| = 1$. Let T_0 be the tetrahedron with vertices

$$(1, 0); \qquad (\omega, 0); \qquad \left(0, \frac{1}{\sqrt{3}}\right); \qquad \left(0, \frac{-1}{\sqrt{3}}\right).$$

One checks easily that all points of T_0 are $2/\sqrt{3}$ units apart, so that T_0 really is a regular tetrahedron.

Consider the new tetrahedron T_n, with vertices

$$(\omega^n, 0); \qquad (\omega^{n+1}, 0); \qquad \left(0, \frac{1}{\sqrt{3}}\right); \qquad \left(0, \frac{-1}{\sqrt{3}}\right).$$

The tetrahedra $T_0, T_1, T_2, \ldots$ are just rotated copies of T_0. We are rotating about the vertical axis. Notice that T_{n+1} and T_n share a face for every n. To say that the dihedral angle is irrational is the same as saying that the list $T_0, T_1, T_2, \ldots$ is infinite. This is the same as saying that there is no n such that $\omega^n = 1$.

In the next section, we will rule out the possibility that $\omega^n = 1$ for any positive integer n. This means that $T_0, T_1, T_2 \ldots$ really is an infinite list. Hence, the common dihedral angle associated to the edges of a regular tetrahedron is irrational.

3 Irrationality Proof

The point of this section is to prove the following result: The complex number

$$\omega = \frac{1}{3} + \frac{2\sqrt{2}}{3} i. \tag{3}$$

does not satisfy the equation $\omega^n = 1$ for any positive integer n. We check by hand the cases $n = 1, 2, 3, 4, 5, 6$, leaving the case $n \geq 7$.

Let $G(\omega)$ be the set of numbers of the form $a + b\omega$, where a and b are integers. This set is discrete: every disk intersects only finitely many elements of $G(\omega)$.

Let $n \geq 7$ be the smallest value such that (supposedly) $\omega^n = 1$. Let $\boldsymbol{Z}[\omega]$ denote the set of numbers of the form

$$a_1\omega + a_2\omega^2 + ... + a_n\omega^n \tag{4}$$

where $a_1, ..., a_n$ are integers. $\boldsymbol{Z}[\omega]$ has the nice property that

$$(\omega^a - \omega^b)^c \in \boldsymbol{Z}[\omega] \tag{5}$$

for any positive integers a, b, c. This comes from the fact that $\omega^n = 1$. There are at least 7 powers of ω crowded on the unit circle, so at least 2 of them must be closer then 1 unit apart. But that means we can find integers a and b such that $|z| < 1$, where $z = \omega^a - \omega^b$. The numbers $z, z^2, z^3...$ all belong to $\boldsymbol{Z}[\omega]$, and these numbers are distinct because $|z^{n+1}| = |z||z^n| < |z^n|$. So, $\boldsymbol{Z}[\omega]$ is not discrete.

We check that ω satisfies $\omega^2 = (2/3)\omega - 1$. From this equation, we get

$$\omega^3 = \omega \times \omega^2 = \omega \times ((2/3)\omega - 1) = (2/3)\omega^2 - \omega = (5/9)\omega - (2/3),$$

and similarly for higher powers of ω. In general,

$$3^n(a_1\omega + ... + a_n\omega^n) = \text{integer} + \text{integer} \times \omega. \tag{6}$$

for any choice of integers $a_1, ..., a_n$. But then $\boldsymbol{Z}[\omega]$ is contained in a scaled-down copy of $G(\omega)$, and hence is discrete. But $\boldsymbol{Z}[\omega]$ is not discrete, and we have a contradiction.

4 Rational Vector Spaces

Let $\mathcal{R} = \{r_1, ..., r_n\}$ be a finite list of real numbers. Let V be the set of all numbers of the form

$$a_0 + a_1 r_1 + ... + a_N r_N; \qquad a_0, a_1, ..., a_N \in \boldsymbol{Q}.$$

V is a finite dimensional $\boldsymbol{Q}$-vector space.

We declare two elements $v_1, v_2 \in V$ to be *equivalent* if $v_1 - v_2 \in \boldsymbol{Q}$. In this case we write $v_1 \sim v_2$. Let $[v]$ denote the set of all elements of V that are equivalent to v. Let W denote the set of equivalence classes of V. The two operations are given by

$$[v] + [w] = [v + w]; \qquad r[v] = [rv].$$

The 0-element is given by $[0]$. One checks easily that these definitions make sense, and turn W into another finite-dimensional $\boldsymbol{Q}$-vector space.

Let $v_1, ..., v_m$ be a basis for V and let $w_1, ..., w_n$ be a basis for W. The *tensor product* $V \otimes W$ is the $\boldsymbol{Q}$-vector space of formal linear combinations

$$\sum_{i,j} a_{ij}(v_i \otimes w_j); \qquad a_{ij} \in \boldsymbol{Q} \tag{7}$$

Here $v_i \otimes w_j$ is just a formal symbol, but in a compatible way the symbol $\otimes$ defines a bilinear map from $V \times W$ into $V \otimes W$:

$$\left(\sum a_i v_i\right) \otimes \left(\sum b_j w_j\right) = \sum a_i b_j (v_i \otimes w_j). \tag{8}$$

The $m \times n$ elements $\{1(v_i \otimes w_j)\}$ serve as as a basis for $V \otimes W$.

Here is a basic property of $V \otimes W$. If $v \in V$ is nonzero and $w \in W$ is nonzero, then $v \otimes w$ is nonzero. One sees this simply by writing v and w out in a basis, and considering Equation 8. At least one product $a_i b_j$ will be nonzero. In particular

$$6 \otimes \delta \neq 0, \tag{9}$$

where δ is the dijedral angle of the regular tetrahedron, and $\mathcal{R}$ is chosen so as to contain δ.

5 Dehn's Invariant

Let $\mathcal{R} = \{r_1, ..., r_N\}$ be a finite list of real numbers, and let V and W be the two examples of vector spaces given in Examples 1 and 2 above. Once again, V is the set of all numbers of the form

$$a_0 + a_1 r_1 + ... + a_n r_N; \qquad a_0, ..., a_N \in \boldsymbol{Q},$$

and W is the set of equivalence classes in V.

Suppose that X is a polyhedron. Let $\lambda_1, ..., \lambda_k$ denote the side lengths of all the edges of X. Let $\theta_1, ..., \theta_k$ be the dihedral angles, listed in the same order. We say that X is *adapted* to $\mathcal{R}$ if

$$\lambda_1, ..., \lambda_k, \theta_1, ..., \theta_k \in \mathcal{R}. \tag{10}$$

If X is adapted to $\mathcal{R}$, we define the *Dehn invariant* as:

$$\langle X \rangle = \sum_{i=1}^{k} (\lambda_i \otimes [\theta_i]) \in V \otimes W \tag{11}$$

The operation $\otimes$ is as in Equation 7, and the addition makes sense because $V \otimes W$ is a vector space.

Suppose now that P and Q are a cube and a regular tetrahedron having the same volume. Assume $\mathcal{R}$ is chosen large enough so that P and Q are both adapted to $\mathcal{R}$. Let λ_P and λ_Q denote the side lengths of P and Q respectively. Let δ_P and δ_Q denote the respective dihedral angles. We have $[\delta_P] = [1/4] = [0]$, because $1/4$ is rational. On the other hand, we have already seen that δ_Q is irrational. Hence $[\delta_Q] \neq [0]$. This gives us

$$\langle P \rangle = 12\lambda_P \otimes [\delta_P] = [0]; \qquad \langle Q \rangle = 6\lambda_Q \otimes [\delta_Q] \neq [0]. \tag{12}$$

In particular,

$$\langle P \rangle \neq \langle Q \rangle \tag{13}$$

To prove Dehn's theorem, our strategh is to show that the Dehn invariant is the same for two polyhedra that are scissors congruent. The result in the next section is the key step in this argument.

6 Clean Dissections

Say that a *clean dissection* of a polyhedron X is a dissection $X = X_1 \cup ... \cup X_N$, where each pair of polyhedra are either disjoint or share precisely a lower-dimensional face. Let $\mathcal{R}$ be as above.

Lemma 6.1 *Suppose that $X = X_1 \cup ... \cup X_N$ is a clean dissection, and all polyhedra are adapted to $\mathcal{R}$. Then $\langle X \rangle = \langle X_1 \rangle + ... + \langle X_N \rangle$.*

Proof: We will let Y stand for a typical polyhedron on our list. Say that a *flag* is a pair (e, Y), where e is an edge of Y. Then

$$\langle X_1 \rangle + ... + \langle X_N \rangle = S = \sum_{\text{flags}} \lambda \otimes \theta'$$

We classify the flag (e, Y) as one of three types.

Type-1: e does not lie on the boundary of P.

Type-2: e lies in the boundary of P, but not in an edge of P.

Type-3: e lies in an edge of P.

We can write $S = S_1 + S_2 + S_3$ where S_j is the sum over flags of Type j.

Call two flags (e, Y) and (e', Y') *strongly equivalent* iff $e = e'$. Given a Type-1 edge e, let $\theta_1, ..., \theta_m$ denote the dijedral angles associated to the flags involving e. From the clean dissection property, these polyhedra fit exactly around e, so that (with our special units) $\theta_1 + ... + \theta_m = 1$. Hence

$$\sum \lambda(e) \otimes [\theta_j] = \lambda(e) \otimes \sum [\theta_j] = \lambda(e) \otimes [1] = 0.$$

Summing over all Type-1 equivalence classes, we find that $S_1 = 0$. A similar argument shows that $S_2 = 0$. In this case $\theta_1 + ... + \theta_k = 1/2$.

Now we show that $S_3 = \langle X \rangle$. Define a *weak equivalence class* as follows: (e, P) and (e', P') are weakly equivalent iff e and e' lie in the same edge of X. The weak equivalence classes are bijective with the edges of X. Let e be some edge of X, with length and dihedral angle λ and θ. Let $e_1, ..., e_m$ be the edges that appear in weak equivalence class named by e. With the obvious notation $\lambda = \lambda_1 + ... + \lambda_k$. Let $\theta_{j1}, ..., \theta_{jm_j}$ denote the dihedral angles associated to the strong equivalence class involving e_j. We have $\theta_{j1} + ... \theta_{jm_j} = \theta$. Summing over the weak equivalence class, we get

$$\sum_{jk} \lambda_j \otimes [\theta_{jk}] = \sum_j \lambda_j \otimes [\theta] = \lambda(e) \otimes [\theta(e)].$$

Summing over all weak equivalence classes, we get $S_3 = \langle X \rangle$, as desired. ♠

7 The Proof

Let P be a cube and let Q be a tetrahedron. We will suppose that we scissors congruence between $P = P_1 \cup ... \cup P_n$ and $Q = Q_1 \cup ... \cup Q_n$.

We first produce new dissections of P and Q that are clean. Here is the construction. Let $\Pi_1, ..., \Pi_k$ denote the union of all the planes obtained by extending the faces of any polyhedron in the above dissection of P. Say that a *chunk* is the closure of a component of $\boldsymbol{R} - \bigcup \Pi_j$. Then we have clean dissections

$$P_i = P_{i1} \cup ... \cup P_{in_i} \tag{14}$$

of each P_i into chunks, and also the clean dissection

$$P = \bigcup P_{ij} \tag{15}$$

of P into chunks. We make all the same definitions for Q. The dissections in Equation 15 for P and Q might not define a scissors congruence, but we don't care.

Let $\mathcal{R}$ denote the finite list of lengths and dihedral angles that arise in any of the polyhedra appearing in our constructions involving P and Q. Let $V \otimes W$ be the vector space defined as in the previous sections, relative to $\mathcal{R}$. Computing the Dehn invariants in $V \otimes W$, we have

$$\langle P \rangle = \sum \langle P_{ij} \rangle = \sum \langle P_i \rangle = \sum \langle Q_i \rangle = \sum \langle Q_{ij} \rangle = \langle Q \rangle. \tag{16}$$

The first equality is obtained by applying Lemma 6.1 to the dissection in Equation 15. The second equality is obtained by applying Lemma 6.1 to each dissection in Equation 14 and adding the results. The middle equality comes from the obvious isometric invariance of the Dehn invariant. The last two equalities have the same explanations as the first two. In short,

$$\langle P \rangle = \langle Q \rangle. \tag{17}$$

This contradicts out computation that $\langle P \rangle \neq \langle Q \rangle$. The only way out of the contradiction is that the cube and the tetrahedron are not scissors congruent.