Complex Hyperbolic Triangle Groups

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1 Introduction

A basic problem in geometry is the *deformation problem*. One starts with a finitely generated group Γ, a Lie group G_1, and a larger Lie group $G_2 \supset G_1$. Given a discrete embedding $\rho_0 : \Gamma \to G_1$ one asks if ρ_0 fits inside a family $\rho_t : \Gamma \to G_2$ of discrete embeddings. Here *discrete embedding* means an injective homomorphism onto a discrete set.

A nice setting for the deformation problem is the case when G_1 and G_2 are isometry groups of rank one symmetric spaces, X_1 and X_2, and Γ is isomorphic to a lattice in G_1. If $X_1 = \boldsymbol{H}^2$, the hyperbolic plane, and $X_2 = \boldsymbol{H}^3$, hyperbolic 3-space, then we are dealing with the classic and well-developed theory of quasifuchsian groups.

The (p,q,r)-*reflection triangle group* is possibly the simplest kind of lattice in $\text{Isom}(\boldsymbol{H}^2)$. This group is generated by reflections in the sides of a geodesic triangle having angles π/p, π/q, π/r (subject to the inequality $1/p+1/q+1/r < 1$.) We allow the possibility that some of the integers are infinite. For instance, the $(2,3,\infty)$-reflection triangle group is commensurable to the classical modular group.

The reflection triangle groups are rigid in $\text{Isom}(\boldsymbol{H}^3)$, in the sense that any two discrete embeddings of the same group are conjugate. We are going to replace $\boldsymbol{H}^3$ by $\boldsymbol{CH}^2$, the complex hyperbolic plane. In this case, we get nontrivial deformations. These deformations provide an attractive problem, because they furnish some of the simplest interesting examples in the still mysterious subject of complex hyperbolic deformations. While some progress has been made in understanding these examples, there is still a lot unknown about them.

In §2 we will give a rapid introduction to complex hyperbolic geometry. In §3 we will explain how to generate some complex hyperbolic triangle groups. In §4 we will survey some results about these groups and in §5 we will present a more complete conjectural picture. In §6 we will indicate some of the techniques we used in proving our results.

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2 The Complex Hyperbolic Plane

The book [**8**]is an excellent general reference for complex hyperbolic geometry. Here are some of the basics.

$\boldsymbol{C}^{2,1}$ is a copy of the vector space $\boldsymbol{C}^3$ equipped with the Hermitian form

$$\langle U, V\rangle = -u_3\overline{v}_3 + \sum_{j=1}^{n} u_j\overline{v}_j \tag{1}$$

Here $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$. A vector V is called *negative*, *null*, or *positive* depending (in the obvious way) on the sign of $\langle V, V\rangle$. We denote the set of negative, null, and positive vectors, by N_-, N_0 and N_+ respectively.

$\boldsymbol{C}^2$ includes in complex projective space $\boldsymbol{CP}^2$ as the affine patch of vectors with nonzero last coordinate. Let $[\,]: \boldsymbol{C}^{2,1} - \{0\} \to \boldsymbol{CP}^2$ be the projectivization whose formula, expressed in the affine patch, is

$$[(v_1, v_2, v_3)] = (v_1/v_3, v_2/v_3). \tag{2}$$

The *complex hyperbolic plane*, $\boldsymbol{CH}^2$, is the projective image of the set of negative vectors in $\boldsymbol{C}^{2,1}$. That is, $\boldsymbol{CH}^2 = [N_-]$. The ideal boundary of $\boldsymbol{CH}^2$ is the unit sphere $S^3 = [N_0]$. If $[X], [Y] \in \boldsymbol{CH}^n$ the complex hyperbolic distance $\varrho([X], [Y])$ satisfies

$$\varrho([X], [Y]) = 2\cosh^{-1}\sqrt{\delta(X, Y)}; \qquad \delta(X, Y) = \frac{\langle X, Y\rangle\langle Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle}. \tag{3}$$

Here X and Y are arbitrary lifts of $[X]$ and $[Y]$. See [**8**, p. 77]. The distance we defined is induced by an invariant Riemannian metric of sectional curvature pinched between -1 and -4. This Riemannian metric is the real part of a Kaehler metric.

$SU(2,1)$ is the Lie group of $\langle,\rangle$ preserving complex linear transformations. $PU(2,1)$ is the projectivization of $SU(2,1)$ and acts isometrically on $\boldsymbol{CH}^2$. The map $SU(2,1) \to PU(2,1)$ is a 3-to-1 Lie group homomorphism. The group of holomorphic isometries of $\boldsymbol{CH}^2$ is exactly $PU(2,1)$. The full group of isometries of $\boldsymbol{CH}^2$ is generated by $PU(2,1)$ and by the antiholomorphic map $(z_1, z_2, z_3) \to (\overline{z}_1, \overline{z}_2, \overline{z}_3)$.

An element of $PU(2,1)$ is called *elliptic* if it has a fixed point in $\boldsymbol{CH}^2$. It is called *hyperbolic* (or *loxodromic*) if there is some $\epsilon > 0$ such that every point in $\boldsymbol{CH}^2$ is moved at least ϵ by the isometry. An element which is neither elliptic nor hyperbolic is called *parabolic*.

$\boldsymbol{CH}^2$ has two different kinds of totally geodesic subspaces, *real slices* and *complex slices*. Every real slice is isometric to $\boldsymbol{CH}^2 \cap \boldsymbol{R}^2$ and every complex slice is isometric to $\boldsymbol{CH}^2 \cap \boldsymbol{C}^1$. The ideal boundaries of real and complex slices are called, respectively, $\boldsymbol{R}$-circles and $\boldsymbol{C}$-circles. The complex slices naturally implement the Poincaré model of the hyperbolic plane and the real slices naturally model the Klein model. It is a beautiful feature of the complex hyperbolic plane that it contains both models of the hyperbolic plane.

3 Reflection Triangle Groups

There are two kinds of reflections in $\text{Isom}(\boldsymbol{CH}^2)$. A *real reflection* is an anti-holomorphic isometry conjugate to the map $(z,w)\to(\overline{z},\overline{w})$. The fixed point set of a real reflection is a real slice. We shall not have much to say about the explicit computation of real reflections, but rather will concentrate on the complex reflections.

A *complex reflection* is a holomorphic isometry conjugate to the involution $(z,w)\to(z,-w)$. The fixed point set of a complex reflection is a complex slice. There is a simple formula for the general complex reflection: Let $C\in N_+$. Given any $U\in \boldsymbol{C}^{2,1}$ define

$$I_C(U)=-U+\frac{2\langle U,C\rangle}{\langle C,C\rangle}C. \tag{4}$$

I_C is a complex reflection.

We also have the formula

$$U\boxtimes V=(\overline{u_3v_2-u_2v_3},\ \overline{u_1v_3-u_3v_1},\ \overline{u_1v_2-u_2v_1}). \tag{5}$$

This vector is such that $\langle U,U\boxtimes V\rangle=\langle V,U\boxtimes V\rangle=0$. See [**8**, p. 45].

Equations 4 and 5 can be used in tandem to rapidly generate triangle groups defined by complex reflections. One picks three vectors $V_1,V_2,V_2\in N_-$. Next, we let $C_j=V_{j-1}\boxtimes V_{j+1}$. Indices are taken mod 3. Finally, we let $I_j=I_{C_j}$. The complex reflection I_j fixes the complex line determined by the points $[V_{j-1}]$ and $[V_{j+1}]$. This, the group $\langle I_1,I_2,I_3\rangle$ is a complex-reflection triangle group determined by the triangle with vertices $[V_1]$, $[V_2]$, $[V_3]$.

Here is a quick dimension count for the space of (p,q,r)-triangle groups generated by complex reflections. We can normalize so that $[V_1]=0$. The stabilizer of 0 in $PU(2,1)$ acts transitively on the unit tangent space at 0. We can therefore normalize so that $[V_2]=(s,0)$ where $s\in(0,1)$. Finally, the isometries $(z,w)\to(z,\exp(i\theta)w)$ stabilize both $[V_1]$ and $[V_2]$. Applying a suitable isometry we arrange that $[V_3]=(t+iu,v)$ where $t,u,v\in(0,1)$. We cannot make any further normalizations, so the space of triangles in $\boldsymbol{CH}^2$ mod isometry is 4-real dimensional. Each of the three angles (p,q,r) puts 1 real constraint on the triangle. For instance, the p-angle places the constraint that $(I_1I_2)^p$ is the identity. Since $4-3=1$, we see heuristically that the space of (p,q,r)-complex reflection triangle groups is 1-real dimensional.

The argument we just gave can be made rigorous, and extends to the case when some of the integers are infinite. (In this case the corresponding vectors are null rather than negative.) In the (∞,∞,∞)-case, the parameter is the *angular invariant* $\arg(\langle V_1,V_2\rangle\langle V_2,V_3\rangle\langle V_3,V_1\rangle)$. Compare [**10**].

This 1-dimensionality of the deformation space makes the (p,q,r)-triangle groups an especially attractive problem to study. Indeed, there is a completely canonical path of deformations. The starting point for the path of deformations is the case when the vectors have entirely real entries. (That is, $u=0$.) In this case, the three complex reflections stabilize the real slice $\boldsymbol{R}^2\cap\boldsymbol{CH}^2$.

4 Some Results

To obtain a deformation of the (p, q, r)-reflection triangle group we choose a slice, either real or complex, and a triple of reflections, either real or complex, which restrict to the reflections in the sides of a (p, q, r)-geodesic triangle in the slice. *A priori* there are 4 possibilities, given that the slice and the reflection types can be either real or complex. These choices lead to different outcomes.

If we start with complex reflections stabilizing a complex slice, the group has order 2, because the reflections will all stabilize the same slice.

A more interesting case involving complex slices is given by:

Theorem 4.1 (8) *$\rho_0 : \Gamma \to Isom(\boldsymbol{CH}^2)$ stabilizes a complex slice and acts on this slice with compact quotient then any nearby representation ρ_t also stabilizes a complex slice.*

Goldman's theorem applies to any co-compact lattice $\rho_0(\Gamma)$. In the case of triangle groups, which are rigid in $\boldsymbol{H}^2$, it says that any nearby representation is conjugate the original. In contrast:

Theorem 4.2 (4, 12) *There is a 1-parameter family $\rho_t(\Gamma(2, 3, \infty))$ of discrete faithful representations of the modular group having the property that ρ_0 stabilizes a real slice and ρ_1 stabilizes a complex slice. For every parameter the generators are real reflections.*

Thus, in the case of non-cocompact triangle groups, two of the remaining 3 cases can be *connected*. In their paper, Falbel and Koseleff claim that their technique works for $\Gamma(p, q, \infty)$ when $\max(p, q) = 4$. For higher values of p and q it is not known what happens.

The remaining case occurs when we start with complex reflections stabilizing a real slice. This is the case we discussed in the previous section. Henceforth we restrict our attention to this case.

Goldman and Parker introduced this topic and studied the case of the ideal triangle group $\Gamma(\infty, \infty, \infty)$. They found that there is a 1-real parameter family of non-conjugate representations, $\{\rho_t,\ t \in (-\infty, \infty)\}$. Once again ρ_0 stabilizes a real slice. Paraphrasing their more precise formulation:

Theorem 4.3 (10) *There are symmetric neighborhoods $I \subset J$ of 0 such that ρ_t is discrete and faithful if $t \in I$ and not both discrete and faithful if $t \notin J$.*

J consists of the the parameter values t such that the element $\rho_t(I_1 I_2 I_3)$ is not an elliptic element. For $t \notin J$, this element is elliptic. If it has finite order then the representation is not faithful; if it has infinite order then the representation is not discrete. The (very slightly) smaller interval I is the inverval for which their proof works. They conjectured that ρ_t should be discrete and faithful iff $t \in J$.

We proved the Goldman-Parker conjecture, and sharpened it a bit.

Theorem 4.4 (16) *ρ_t is discrete and faithful if and only if $t \in J$. Furthermore, ρ_t is indiscrete if $t \notin J$.*

The group $L = \rho_s(\Gamma(\infty, \infty, \infty))$, when $s \in \partial J$ is especially beautiful. We call this group the *last ideal triangle group*. (There are really two groups, one for each endpoint of J, but these are conjugate.) This group seems central in the study of complex hyperbolic deformations of the modular group. For instance, Falbel and Parker recently discovered that L arises as the endpoint of a certain family of deformations of the modular group, using real reflections. See [**5**] for details.

Recall that L, like all discrete groups, has a *limit set* $\Omega(L) \subset S^3$ and a *domain of discontinuity* $\Delta(L) = S^3 - \Omega(L)$. The quotient $\Delta(L)/L$ is a 3-dimensional orbifold, commonly called the *orbifold at infinity*.

Theorem 4.5 (17) *$\Delta(L)/L$ is commensurable to the Whitehead link complement.*

The Whitehead link complement is a classic example of a finite volume hyperbolic 3-manifold. The surprise in the above result is that a real hyperbolic 3-manifold makes its appearance in the context of complex hyperbolic geometry.

One might wonder about analogues of Theorem 4.4 for other triangle groups. Below we will conjecture that the space of discrete embeddings is a certain interval. In his thesis [**22**], Justin Wyss-Gallifent studied some special cases of this question. He made a very interesting discovery concerning the $(4, 4, \infty)$ triangle group:

Theorem 4.6 (22) *Let S be the set of parameters t for which the representation $\rho_t(\Gamma(4, 4, \infty))$ is discrete (but not necessarily injective). Then S contains isolated points and, in particular, is not an interval.*

There seems to be an interval J of discrete embeddings and, outside of J, an extra countable sequence $\{t_j\}$ of parameters for which ρ_{t_j} is discrete but not an embedding. This sequence accumulates on the endpoints of J.

Motivated by [**17**] I wanted to produce a discrete complex hyperbolic group whose orbifold at infinity was a closed hyperbolic 3-manifold. The extra representations found by Wyss-Gallifent seemed like a good place to start. Unfortunately, there is a cusp built into the representations of the $(4, 4, \infty)$ triangle groups.

Instead, I considered the $(4, 4, 4)$-groups, and found that the extra discrete deformations exist. $\rho_t(\Gamma(4, 4, 4))$ seems to be discrete embedding iff all the elements of the form $\rho_t(I_iI_jI_iI_k)$ are not elliptic. Here i, j, k are meant to be distinct. (For all these parameters, the element $\rho_t(I_iI_jI_k)$ is still a loxodromic element.) There is a countable collection $t_5, t_6, ...$ of parameters such that $\rho_{t_j}(I_iI_jI_iI_k)$ has order j. All these representations seem discrete. For ease of notation we set $\rho_j = \rho_{t_j}$.

For $j = 5, 6, 7, 8, 12$ we can show by arithmetic means that ρ_j is discrete. The representation ρ_5 was too complicated for me to analyze and ρ_6 has a cusp. The simplest remaining candidate is ρ_7.

Theorem 4.7 (18) *$G = \rho_7(\Gamma(4,4,4))$ is a discrete group. The orbifold at infinity $\Delta(G)/G$ is a closed hyperbolic 3-orbifold.*

In the standard terminology, $\Delta(G)/G$ is the orbifold obtained by labelling the braid $(AB)^{15}(AB^{-2})^3$ with a 2. Here A and B are the standard generators of the 3-strand braid group.

A *spherical CR structure* on a 3-manifold is a system of coordinate charts into S^3 whose transition functions are restrictions of complex projective transformations. Kamishima and Tsuboi [**13**] produced examples of spherical CR structures on Seifert fibered 3-manifolds, but our example in theorem 4.7 gives the only known spherical CR structure on a closed hyperbolic 3-manifold. We think that Theorem 4.7 holds for all $j = 8, 9, 10....$

Concerning the specific topic of triangle groups generated by complex reflections, I think that not much else is known. Recently a lot of progress has been made in understanding triangle groups generated by real reflections. See [**3**] and [**4**]. There has been a lot of other great work done recently on complex hyperbolic discrete groups, for instance [**1**], [**2**], [**9**], [**20**], [**21**]. Also see the references in Goldman's book [**8**].

5 A Conjectural Picture

We will consider the 1-parameter family $\rho_t(p,q,r)$ of representations of the (p,q,r)-reflection triangle group, using complex reflections. We arrange that ρ_0 stabilizes a real slice. We choose our integers so that $p \le q \le r$. We let I_p, I_q, I_r be the generators of the reflection triangle group. The notation is such that I_p is the reflection in the side of the triangle opposite p, etc. Define

$$W_A = I_p I_r I_q I_r; \qquad W_B = I_p I_q I_r. \tag{6}$$

Conjecture 5.1 *The set of t for which $\rho_t(p,q,r)$ is a discrete embedding is the closed interval consisting of the parameters t for which neither $\rho_t(W_A)$ nor $\rho_t(W_B)$ is elliptic.*

We call the interval of Conjecture 5.1 the *critical interval*.

We say that the triple (p,q,r) has *type A* if the endpoints of the critical interval correspond to the representations when W_A is a parabolic element. In other words, W_A becomes elliptic before W_B. We say otherwise that (p,q,r) has *type B*.

Conjecture 5.2 *The triple (p,q,r) has type A if $p < 10$ and type B if $p > 13$.*

The situation is rather complicated when $p \in \{10, 11, 12, 13\}$. Our Java applet [**19**] lets the user probe these cases by hand, though the roundoff error makes a few cases ambiguous. The extra deformation, which was the subject of Theorem 4.7, seems part of a more general pattern.

Conjecture 5.3 *If (p,q,r) has type A then there is a countable collection of parameters $t_1, t_2, t_3...$ for which $\rho_{t_j}(p,q,r)$ is infinite and discrete but not injective. If (p,q,r) has type B then all infinite discrete representations $\rho_t(p,q,r)$ are embeddings and covered by Conjecture 5.1.*

The *proviso* about the infinite image arises because there always exists an extremely degenerate representation of $\Gamma(p,q,r)$ onto $\boldsymbol{Z}/2$. The generators are all mapped to the same complex reflection.

In summary, there seems to be a critical interval I, such the representations $\rho_t(p,q,r)$ are discrete embeddings iff $t \in I$. Depending on the endpoints of I, there are either no additional discrete representations, or a countable collection of extra discrete representations.

It is interesting to see what happens as t moves to the boundary of I from within I. We observed a certain kind of monotonicity to the way the representation varies. Let Γ be the abstract (p,q,r) triangle group. For any word $W \in \Gamma$, let $W_t = \rho_t(W)$. We will concentrate on the case when W is an infinite word. For $t \in I$, the element W_t is (conjecturally) either a parabolic or loxodromic. Let $\lambda(W_t)$ be the translation length of W_t.

Conjecture 5.4 *As t increases monotonically from 0 to ∂I, the quantity $\lambda(W_t)$ decreases monotonically for all infinite words W.*

Conjecture 5.4 is closely related to some conjectures of Hanna Sandler [**15**] about the behavior of the trace function in the ideal triangle case. I think that there is some fascinating algebra hiding behind the triangle groups–in the form of the behavior of the trace function–but so far it is unreachable.

6 Some Techniques of Proof

If $G \subset$Isom(X), one can try to show that G is discrete by constructing a *fundamental domain* for G. One looks for a set $F \subset X$ such that the orbit $G(F)$ *tiles* X. This means that the translates of F only intersect F in its boundary. The Poincaré theorem [**B**, §9.6] gives a general method for establishing the tiling property of F based on how certain elements of G act on ∂F.

When $X = \boldsymbol{H}^n$, one typically builds fundamental domains out of polyhedra bounded by totally geodesic codimension-1 faces. When $X = \boldsymbol{CH}^n$, the situation is complicated by the absence of totally geodesic codimension-1 subspaces. The most natural replacement is the *bisector*. A bisector is the set of points in $\boldsymbol{CH}^n$ equidistant between two given points. Mostow [**14**] used bisectors in his analysis of some exceptional non-arithmetic lattices in Isom$(\boldsymbol{CH}^2)$, and Goldman studied them extensively in [**8**]. (See Goldman's book for additional references on papers which use bisectors to construct fundamental domains.)

My point of view is that there does not seem to be a "best" kind surface to use in constructing fundamental domains in complex hyperbolic space. Rather, I think that one should be ready to fabricate new kinds of surfaces to fit the problem at hand. It seems that computer experimentation often reveals a good choice of surface to use. In what follows I will give a quick tour of constructive techniques.

Consider first the deformations $G_t = \rho_t(\infty,\infty,\infty)$ of the ideal triangle group, introduced in [**10**]. According to [**16**] these groups are discrete for $t \in [0,\tau]$. Here τ is the *critical parameter* where the product of the generators is parabolic. It is convenient to introduce the *Clifford torus*. Thinking of $\boldsymbol{CH}^2$ as the open unit ball in $\boldsymbol{C}^2$, the Clifford torus is the subset $T = \{|z| = |w|\} \subset S^3$. Amazingly T has 3 foliations by $\boldsymbol{C}$-circles: The *horizontal foliation* consists of $\boldsymbol{C}$-circles of the form $\{(z,w)|z = z_0\}$. The *vertical foliation* consists of $\boldsymbol{C}$-circles of the form $\{(z,w)|w = w_0\}$. The *diagonal foliation* consists of $\boldsymbol{C}$-circles having the form $\{(z,w)|\ z = \lambda_0 w\}$.

Recall that G_t is generated by 3 complex reflections. Each of these reflections fixes a complex slice and hence the bounding $\boldsymbol{C}$-circle. One can normalize so that the three fixed $\boldsymbol{C}$-circles lie on the Clifford torus, one in each of the foliations. Passing to an index 2 subgroup, we can consider a group generated by 4 complex reflections: Two of these reflections, H_1 and H_2, fix horizontal $\boldsymbol{C}$-circles h_1 and h_2 and the other two, V_1 and V_2, fix vertical $\boldsymbol{C}$-circles v_1 and v_2.

The ideal boundary of a bisector is called a *spinal sphere.* This is an embedded 2-sphere which is foliated by $\boldsymbol{C}$-circles (and also by $\boldsymbol{R}$-circles.) We can find a configuration of 4-spinal spheres $S(1,v)$, $S(2,v)$, $S(1,h)$ and $S(2,h)$. Here $S(j,v)$ contains v_j as part of its foliation and $S(j,h)$ contains h_j as part of its foliation. The map H_j stabilizes $S(j,h)$ and interchanges the two components of $S^3 - S(j,h)$. Analogous statements apply to the Vs.

The two spheres $S(h,j)$ are contained in the closure of one one component of $S^3 - T$ and the two spheres $S(v,j)$ are contained in the closure of the other. When the parameter t is close to 0 these spinal spheres are all disjoint from each other, excepting tangencies, and form a kind of necklace of spheres. Given the way the elements H_j and V_j act on our necklace of spheres, we see that we are dealing with the usual picture associated to a *Schottky group*. In this case the discreteness of the group is obvious.

As the parameter increases, the two spinal spheres $S(v,1)$ and $S(v,2)$ collide. Likewise, $S(h,1)$ and $S(h,2)$ collide. Unfortunately, the collision parameter occurs before the critical parameter. For parameters larger than this collision parameter, we throw out the spinal spheres and look at the action of G on the Clifford torus itself. (This is not the point of view taken in [**10**] but it is equivalent to what they did.)

Let H be the subgroup generated by the reflections H_1 and H_2. One finds that the orbit $H(T)$ consists of translates of T which are disjoint from each other except for forced tangencies. Even though H is an infinite group, most of the elements in H move T well off itself, and one only needs to take care in checking a short finite list of words in H. Once we know how H acts on T we invoke a variant of the *ping-pong lemma* to get the discreteness.

At some new collision parameter, the translates of the Clifford torus collide with each other. Again, the collision parameter occurs before the critical parameter. This is where the work in [**16**] comes in. I define a new kind of surface called a *hybrid cone*. A hybrid cone is a certain surface foliated by arcs of $\boldsymbol{R}$-circles. These arcs make the pattern of a fan: Each arc has one endpoint on the arc of a $\boldsymbol{C}$-circle and the other endpoint at a single point common to all the arcs. I cut out two triangular patches on the Clifford torus and replace each patch by a union of three hybrid cones. Each triangular patch is bounded by three arcs of $\boldsymbol{C}$-circles; so that the hybrid cones are formed by connecting these exposed arcs to auxilliary points using arcs of $\boldsymbol{R}$-circles. In short, I put some dents into the Clifford torus to make it fit better with its H-translates, and the I apply the ping-pong lemma to the dented torus.

I also use hybrid cones in [**17**], to construct a natural fundamental domain in the domain of discontinuity $\Delta(L)$ for the last ideal triangle group L. In this case, the surfaces fit together to make three topological spheres, each tangent to the other two along arcs of $\boldsymbol{R}$-circles. The existence of this fundamental domain lets me compute explicitly that $\Delta(L)/L$ is commensurable to the Whitehead link complement.

Falbel and Zocca [**6**] introduce related surfaces called $\boldsymbol{C}$-spheres, which are foliated by $\boldsymbol{C}$-circles. These surfaces seem especially well adapted to groups generated by real reflections. See [**3**] and [**4**]. Indeed, Falbel and Parker construct a different fundamental domain for L using $\boldsymbol{C}$-spheres. See [**5**].

To prove Theorem 4.7 in [**18**] I introduce another method of constructing fundamental domains. My proof revolves around the construction of a simplicial complex $Z \subset \boldsymbol{C}^{2,1}$. The vertices of Z are canonical lifts to $\boldsymbol{C}^{2,1}$ of fixed points of certain elements of the group $G = \rho_7(\Gamma(4,4,4))$. The tetrahedra of Z are Euclidean convex hulls of various 4-element subsets of the vertices. Comprised of infinitely many tetrahedra, Z is invariant under the element $I_2 I_1 I_3$. Modulo this element Z has only finitely many tetrahedra.

Recall that $[\ ]$ is the projectivization map. Let $[Z_0] = [Z] \cap S^3$. I deduce the topology of the orbifold at infinity by studying the topology of $[Z_0]$. To show that my analysis of the topology at infinity is correct, I show that one component F of $\boldsymbol{CH}^2 - [Z]$ has the *tiling property*: The G-orbit of F tiles $\boldsymbol{CH}^2$. Now, Z is an essentially combinatorial object, and it not too hard to analyze the combinatorics and topology of Z in the abstract. The hard part is showing that the map $Z \to [Z]$ is an embedding. Assuming the embedding, the combinatorics and topology of Z are reproduced faithfully in $[Z]$, and I invoke a variant of the Poincaré theorem.

After making some easy estimates, my main task boils down to showing that the projectivization map $[\ ]$ is injective on all pairs of tetrahedra within a large but finite portion of Z. Roughly, I need to check about 1.3 million tetrahedra. The sheer number of checks forces us to bring in the computer. I develop a technique for proving, with rigorous machine-aided computation, that $[\ ]$ is injective on a given pair of tetrahedra.

A novel feature of my work is the use of computer experimentation and computer-aided proofs. This feature is also a drawback, because it only allows for the analysis of examples one at a time. To make this analysis automatic I would like to see a kind of marriage of complex hyperbolic geometry and computation. On the other hand, I would greatly prefer to see some theoretical advances in discreteness-proving which would eliminate the computer entirely.

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